THE DIFFERENTIAL

AND

THE INTEGRAL CALCULUS

AN INTRODUCTION TO

THE DIFFERENTIAL

AND

THE INTEGRAL

CALCULUS

BY

THOMAS HUGH MILLER, B.A. CANTAB.

FORMERLY SCHOLAR OF PETERHOUSE

MATHEMATICAL LECTURER AT BOROUGH ROAD TRAINING COLLEGE

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NOTE

THIS volume is intended to meet the demands of Teachers and Students, who begin the study of the Calculus before Analytical Geometry. It assumes a knowledge of Elementary Algebra, and Trigonometry as far as the properties of plane triangles. The examples include all those on the subjects treated, set in the South Kensington Examinations of recent years.

T. H. M.

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CHAPTER I.

DEFINITIONS.

1. ANY quantity whose value admits of change is called a **Variable**. Such quantities are distinguished from Constant quantities, whose values admit of no change, such as π, e (the base of the Napierian logarithms), the radius of a given circle, etc.

2. A variable whose value depends on that of another is called a **Function** of the latter, which is called the Independent Variable. For example, the following are functions of x:

$$a+x, \quad x^2, \quad \sqrt{x}, \quad \sin x, \quad \log x, \quad a^x.$$

In general, any quantity which contains x will be a function of x; but this is not a universal rule. $a+x-x$ is evidently not a function of x, and $\sec x \cos x$ is independent of x. A general method of determining whether an expression which contains x is a function of x will be given afterwards.

It is usual to denote variable quantities by x, y, z; constant quantities by a, b, c. A function of x is often denoted by the symbols $f(x)$, $F(x)$, $\phi(x)$. If a quantity is a function of two variables x and y, it is denoted by $f(x, y)$, or $F(x, y)$. If a particular value of x is substituted for x in the function, say $x=a$, the result is written $f(a)$, $F(a)$, or $\phi(a)$. If the values a, b, are written for x and y in a function of two variables, the result is written $F(a, b)$.

3. If $f(x)$ be a function of x, such that, when the change in the value of x is taken indefinitely small the consequent change in the value of $f(x)$ is also indefinitely small, then $f(x)$ is said to be a **Continuous Function** of x.

Thus x^n and $\sin x$ are continuous functions of x.

When an indefinitely small change in the value of x produces a change in the value of $f(x)$, which is finite, or indefinitely great, or is from a real value to an impossible value, $f(x)$ is called a **Discontinuous Function** of x. Thus—

$\frac{1}{1-x}$ is discontinuous for the value of x equal to unity. For let $x=1-\frac{h}{2}$, the value of the function is $\frac{2}{h}$. Next, let $x=1+\frac{h}{2}$, the value of the function is $-\frac{2}{h}$. Therefore a change of h in the value of x produces a change $\frac{4}{h}$ in the value of the function. That is, when h becomes very small the change in the function becomes very great.

Similarly $\frac{3a}{\sqrt{4ax-x^2}}$ is continuous from $x=a$ to $x=4a$, but discontinuous when x equals $4a$, because for any value of x greater than $4a$, the denominator of the fraction is impossible.

4. Let $f(x)$ be a function of x, and let a be such a value of x that $f(a+h)$ or $f(a-h)$ can be made to differ from a fixed value A by a quantity which is indefinitely small when h is indefinitely small, then A is called the **Limiting Value** or the **limit** of $f(x)$ when x equals a.

For example, $\left(1+\frac{1}{x}\right)^x$ has no assignable value when x is indefinitely great; but it is shown in algebra that as x

increases, this expression can be made to differ from e, the base of the natural logarithms, by a quantity which can be made as small as we please. Then e is called the Limit of this expression when x is infinite. This is generally written

$$\text{Lt}_{x=\infty}\left(1+\frac{1}{x}\right)^x=e.$$

Similarly $\frac{\sin x}{x}$ is always a proper fraction, but when x differs from zero by an indefinitely small quantity, it is proved in trigonometry that $\frac{\sin x}{x}$ differs from unity by a quantity which is also indefinitely small. Thus

$$\text{Lt}_{x=0}\frac{\sin x}{x}=1.$$

5. If $f(x)$ is a function of an independent variable, x, then the limiting value of the fraction $\frac{f(x+h)-f(x)}{h}$, when h is indefinitely small, is called the **Differential Coefficient** of $f(x)$ with respect to x.

This limit is often expressed by the symbol $f'(x)$. If a single letter, as y, is written for $f(x)$, the limiting value is generally denoted by $\frac{dy}{dx}$.

The quantity h is called the increment of x. If a particular value of x is substituted for x after the limiting value has been found, the result is written $f'(a)$, where a is the given value of x.

6. Since $\frac{f(x+h)-f(x)}{h}$ is the ratio of the increment of $f(x)$ to the increment of x, the **Limiting Value of this ratio,** $f'(x)$ is the **Rate of Increase of** $f(x)$ compared with that of x.

If $f(x)$ increases as x increases, $f(x+h)$ is greater than $f(x)$ when h is positive, therefore $f'(x)$ is also positive. If $f(x)$ diminishes as x increases, $f(x+h)$ is less than $f(x)$, and $f'(x)$ is negative.

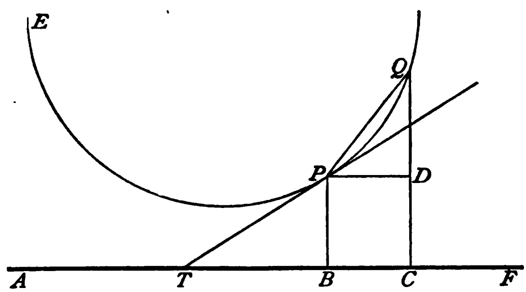
7. If $f(x)$ and $F(x)$ are two functions of x which are always equal to each other, and if when x approaches a, $f(x)$ approaches a limiting value A, and $F(x)$ approaches B, then $A=B$. For $f(x)=A+h$, where h is a quantity that can be made indefinitely small, and $F(x)=B+k$, where k can be made indefinitely small.

$$\text{Then } A+h=B+k,$$

$$\therefore\ A-B=k-h;$$

that is, the difference between A and B can be made less than any finite quantity. The difference must therefore be 0.

8. A geometrical meaning may be assigned to $f'(x)$.



For let EPQ be an arc of a circle; A a fixed point in a given straight line AF; PT the tangent at P. Now if PB,

QC, be drawn perpendicular to AF, from any points P, Q in the arc, the lengths of the lines PB, QC depend on the distances AB, AC; or, if $AB=x$, we may write

$$PB=f(x).$$

Let $BC=h$, and draw PD parallel to AF, and join PQ.

$$\text{Then } QC=f(AC)=f(x+h);$$

$$\therefore \frac{f(x+h)-f(x)}{h}=\frac{QD}{PD}=\tan QPD.$$

Now as h diminishes, the direction of PQ approaches as a limit the line PT;

$$\therefore \text{Lt}\,\frac{f(x+h)-f(x)}{h}=\text{Lt}\tan QPD,$$

$$\text{or} \qquad f'(x) \quad =\tan PTF.$$

CHAPTER II.

DIFFERENTIAL COEFFICIENTS.

To find the Differential Coefficients of the Elementary Functions.

1. cx^n, where n and c are constant.

Let $y=cx^n$, let x become $x+h$, and in consequence let y become $y+k$;

$$\therefore \; y+k=c(x+h)^n;$$
$$\therefore \quad k=c(x+h)^n-cx^n.$$

Now if h is less than x, $(x+h)^n$ can be expanded in a convergent series in descending powers of x by the binomial theorem;

$$\therefore \; k=c[x^n+nx^{n-1}h+(\text{terms involving higher powers of } h)-x^n];$$
$$\therefore \; \frac{k}{h}=cnx^{n-1}+\text{terms in } h,\ h^2, \ . \ . \ . \text{ etc.}$$

But the limiting value of the terms after the first is zero, when h becomes small, and the limiting value of $\frac{k}{h}$ is $\frac{dy}{dx}$;

$$\therefore \; \frac{dy}{dx}=cnx^{n-1}. \quad \text{(See § 7, p. 4.)}$$

(See § 7, p. 4.)

EXAMPLES:

$$\text{If } y=x^2, \quad \frac{dy}{dx}=2x,$$
$$y=\sqrt{x}, \quad \frac{dy}{dx}=\frac{1}{2\sqrt{x}},$$
$$y=\frac{1}{x}, \quad \frac{dy}{dx}=-\frac{1}{x^2},$$
$$y=c, \therefore \; n=0 \text{ and } \frac{dy}{dx}=0.$$

This last result is important, for if an expression is independent of x, its differential coefficient is 0.

2. Let $y=n^x$, where n is constant.

$$\text{Then } y+k=n^{x+h};$$
$$\therefore\ k=n^{x+h}-n^x,$$
$$=n^x(n^h-1).$$

Expanding by the Exponential Theorem,

$$k=n^x(1+h\log_e n+\text{terms in } h^2,\ h^3,\ \text{etc.}\ -1);$$
$$\therefore\ \frac{k}{h}=n^x(\log_e n+\text{terms in } h,\ h^2\ \ldots\ \text{etc.}),$$

when h becomes small, the limits are equal;

$$\therefore\ \frac{dy}{dx}=n^x\log_e n.$$

Example:

$$\text{If } y=e^x,\ \frac{dy}{dx}=e^x.$$

3. Let $y=\log_a x$, where a is constant.

$$\text{Then } y+k=\log_a(x+h);$$
$$\therefore\ k=\log_a(x+h)-\log_a x,$$
$$=\log_a\frac{x+h}{x}=\log_a\left(1+\frac{h}{x}\right),$$
$$=\log_a e\left(\frac{h}{x}-\text{terms in } h^2,\ h^3\ \ldots\ \text{etc.}\right).$$

And this series is convergent if h is less than x;

$$\therefore\ \frac{k}{h}=\log_a e\left(\frac{1}{x}+\text{terms in } h,\ h^2\ \ldots\ \text{etc.}\right).$$

Now when h becomes small, the limits are equal;

$$\therefore\ \frac{dy}{dx}=\frac{\log_a e}{x}.$$

Example:

$$\text{If } y=\log_e x,\ \frac{dy}{dx}=\frac{1}{x}.$$

4. Let $y=\sin(x+a)$, where a is constant.

Then $y+k=\sin(x+h+a)$;

$$\therefore\ k=\sin(x+h+a)-\sin(x+a),$$

$$=2\cos\left(x+a+\frac{h}{2}\right)\sin\frac{h}{2};$$

$$\therefore\ \frac{k}{h}=\cos\left(x+a+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}.$$

Now when h becomes small, the limiting value of $\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}$ is 1, and of $x+a+\frac{h}{2}$ is $x+a$;

$$\therefore\ \frac{dy}{dx}=\cos(x+a).$$

EXAMPLE:

$$\text{If } y=\cos x=\sin\left(x+\frac{\pi}{2}\right),$$

$$\frac{dy}{dx}=\cos\left(x+\frac{\pi}{2}\right)=-\sin x.$$

5. If $y=\tan(x+a)$, where a is constant,

$$y+k=\tan(x+h+a),$$

$$k=\tan(x+h+a)-\tan(x+a),$$

$$=\frac{\sin(x+h+a)\cos(x+a)-\cos(x+h+a)\sin(x+a)}{\cos(x+h+a)\cos(x+a)}.$$

$$=\frac{\sin h}{\cos(x+h+a)\cos(x+a)},$$

$$\therefore\ \frac{k}{h}=\frac{\frac{\sin h}{h}}{\cos(x+h+a)\cos(x+a)},$$

Therefore when h becomes small,

$$\frac{dy}{dx}=\frac{1}{\cos^2(x+a)}=\sec^2(x+a).$$

EXAMPLE:

$$\text{If } y=\cot x=-\tan\left(x+\frac{\pi}{2}\right),$$

$$\frac{dy}{dx}=-\sec^2\left(x+\frac{\pi}{2}\right)=-\operatorname{cosec}^2 x.$$

6. To find the Differential Coefficient of the Sum of two or more Functions.

Let $y=u\pm v$, where u and v are functions of x.

Let x become $x+h$, and in consequence let y, u, v become $y+k$, $u+l$, and $v+m$ respectively.

$$\text{Then } y+k=u+l\pm(v+m);$$

$$\therefore\ k=l\pm m;$$

$$\therefore\ \frac{k}{h}=\frac{l}{h}\pm\frac{m}{h}.$$

But the limiting value of $\frac{k}{h}$ when h is taken very small is $\frac{dy}{dx}$; of $\frac{l}{h}$ and of $\frac{m}{h}$, $\frac{du}{dx}$ and $\frac{dv}{dx}$; $\therefore\ \frac{dy}{dx}=\frac{du}{dx}\pm\frac{dv}{dx}$.

Therefore the Differential Coefficient of the algebraical sum of two functions equals the algebraical sum of their differential coefficients.

If this is true for r functions, it is true for $r+1$. For if

$$y=u_1+u_2+\ \dots\ +u_r+v,$$

$$\frac{dy}{dx}=\frac{d(u_1+u_2+\ \dots\ +u_r)}{dx}+\frac{dv}{dx},$$

$$=\frac{du_1}{dx}+\frac{du_2}{dx}+\ \dots\ +\frac{du_r}{dx}+\frac{dv}{dx}.$$

But it has been shown true if r equals 1, therefore it is true universally.

EXAMPLES:

$$\text{Let } y=\operatorname{versin} x=1-\cos x;\quad \therefore\ \frac{dy}{dx}=\sin x.$$

$$\text{Let } y=\operatorname{covers} x=1-\sin x;\quad \therefore\ \frac{dy}{dx}=-\cos x.$$

7. To find the Differential Coefficient of the Product of two or more Functions.

Using the same notation,

$$\text{Let } y=uv;$$

$$\therefore\ y+k=(u+l)(v+m),$$

$$=uv+um+vl+lm;$$

$$\therefore\quad k=um+vl+lm;$$

$$\therefore\quad \frac{k}{h}=u\frac{m}{h}+v\frac{l}{h}+l\frac{m}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the limit of $\frac{m}{h}$ is $\frac{dv}{dx}$ and l approaches the value zero, therefore $l\left(\frac{m}{h}\right)$ approaches zero. Therefore when h becomes small, equation (1) becomes

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Thus the differential coefficient is found by multiplying the differential coefficient of each factor by the product of the remaining factors.

For suppose this is true for r factors,

$$\text{Let } y=u_1u_2\ .\ .\ .\ u_rv;$$

$$\therefore\ \frac{dy}{dx}=v\frac{du_1u_2\ .\ .\ .\ u_r}{dx}+u_1u_2\ .\ .\ .\ u_r\frac{dv}{dx},$$

$$=vu_2\ .\ .\ .\ u_r\frac{du_1}{dx}+vu_1u_3\ .\ .\ .\ u_r\frac{du_2}{dx}+\ .\ .\ .\ +u_1u_2\ .\ .\ .\ u_r\frac{dv}{dx}.$$

But it is true when r equals 1, and therefore it is true universally.

EXAMPLES:

$$\frac{dcv}{dx}=c\frac{dv}{dx}, \text{ if } c \text{ is constant.}$$

For $\dfrac{dc}{dx}=0.$

Let $y=x\sin x.$

Here $u=x, \therefore \dfrac{du}{dx}=1;$

$$v=\sin x, \therefore \frac{dv}{dx}=\cos x;$$

$$\therefore \frac{dy}{dx}=x\cos x+\sin x.$$

8. **To find the Differential Coefficient of a Quotient.**

Using the same notation,

$$\text{Let } y=\frac{u}{v};$$

$$\therefore y+k=\frac{u+l}{v+m},$$

$$\therefore k=\frac{u+l}{v+m}-\frac{u}{v};$$

$$=\frac{uv+vl-uv-um}{v^2+vm},$$

$$=\frac{vl-um}{v^2+vm};$$

$$\therefore \frac{k}{h}=\frac{v\dfrac{l}{h}-u\dfrac{m}{h}}{v^2+vm}.$$

But the limiting value of vm is zero when h becomes small therefore taking the limit,

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

EXAMPLES:

(1) Let $y=\tan x=\frac{\sin x}{\cos x}$.

$$\text{Here } u=\sin x, \therefore \frac{du}{dx}=\cos x;$$

$$v=\cos x, \therefore \frac{dv}{dx}=-\sin x;$$

$$\therefore \frac{dy}{dx}=\frac{\cos^2 x-(-\sin^2 x)}{\cos^2 x}=\frac{1}{\cos^2 x}.$$

This result has been previously obtained from the definition. (§ 5, p. 8.)

(2) Let $y=\sec x=\frac{1}{\cos x}$.

$$\text{Here } u=1, \quad \therefore \frac{du}{dx}=0.$$

$$v=\cos x, \therefore \frac{dv}{dx}=-\sin x;$$

$$\therefore \frac{dy}{dx}=\frac{\sin x}{\cos^2 x}.$$

$$\text{And similarly } \frac{d \operatorname{cosec} x}{dx}=-\frac{\cos x}{\sin^2 x}.$$

These results may also be obtained from the definition.

(3) Let $y=\log_x a$,

$$=\frac{1}{\log_a x},$$

$$\therefore \frac{dy}{dx}=-\frac{\frac{\log_a e}{x}}{(\log_a x)^2},$$

$$=-\frac{1}{x\log_e a\,(\log_a x)^2}.$$

9. If $y=f(x)$, the differential coefficient of y with respect to x has been defined as the limiting value of $\frac{k}{h}$, where h is an increment in the value of x, and k the consequent increment in the value of y, when h becomes indefinitely small. Now if any change be made in the value of y, the value of x will in general change—that is, x will be a function of y. Let this value be $F(y)=x$. Now when x became $x+h$, y assumed the value $y+k$. If, therefore, in the equation $F(y)=x$, $y+k$ be written for y, one of the values which x takes must be $x+h$.

Then the limiting value of $\frac{h}{k}$, when k becomes small, will be the differential coefficient of x with respect to y, and is written $\frac{dx}{dy}$.

$$\text{But } \frac{h}{k}\times\frac{k}{h}=1 \text{ always;}$$

therefore, when h and k become small,

$$\frac{dy}{dx}\times\frac{dx}{dy}=1.$$

For example:

$$\text{If } y=x^2, \qquad \frac{dy}{dx}=2x.$$

$$\text{also } x=\pm y^{\frac{1}{2}} \therefore \frac{dx}{dy}=\pm\tfrac{1}{2}y^{-\frac{1}{2}}.$$

Therefore, taking the positive value of $\frac{dx}{dy}$,

$$\frac{dy}{dx}\times\frac{dx}{dy}=2x\,.\,\tfrac{1}{2}y^{-\frac{1}{2}},$$

$$=\frac{x}{\sqrt{y}}=\frac{x}{x}=1.$$

10. **Application to the Inverse Trigonometrical Functions.**

Let $x=a\sin y$, where a is constant.

Then y is an angle whose sine is $\frac{x}{a}$, or as it is written,

$$y=\sin^{-1}\frac{x}{a}.$$

$$\text{But } \frac{dx}{dy}=a\cos y;$$

$$\therefore \frac{dy}{dx}=\frac{1}{a\cos y}=\frac{1}{a\sqrt{1-\sin^2 y}},$$

$$\frac{dy}{dx}=\frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}=\frac{1}{\sqrt{a^2-x^2}}.$$

Corollary:

$$\text{If } y_1=\cos^{-1}\frac{x}{a}, \text{ then } y+y_1=\frac{\pi}{2};$$

$$\therefore \frac{dy}{dx}+\frac{dy_1}{dx}=0;$$

$$\therefore \frac{dy_1}{dx}=-\frac{1}{\sqrt{a^2-x^2}}.$$

11. Next let $y = \tan^{-1}\frac{x}{a}$, or $x = a \tan y$.

Then $\frac{dx}{dy} = a(1 + \tan^2 y)$,

$$\therefore \frac{dy}{dx} = \frac{1}{a(1+\tan^2 y)} = \frac{1}{a\left(1 + \frac{x^2}{a^2}\right)},$$

$$= \frac{a}{x^2 + a^2}.$$

Corollary :

If $y_1 = \cot^{-1}\frac{x}{a}$, then $y + y_1 = \frac{\pi}{2}$;

$$\therefore \frac{dy_1}{dx} = -\frac{dy}{dx} = -\frac{a}{x^2 + a^2}.$$

12. Let $y = \sec^{-1}\frac{x}{a}$, or $x = a \sec y$.

Then $\frac{dx}{dy} = \frac{a \sin y}{\cos^2 y}$,

$$\therefore \frac{dy}{dx} = \frac{\cos^2 y}{a \sin y}.$$

But $\cos y = \frac{a}{x}$;

$$\therefore \frac{dy}{dx} = \frac{\frac{a^2}{x^2}}{a\sqrt{1 - \frac{a^2}{x^2}}} = \frac{a}{x\sqrt{x^2 - a^2}}.$$

And if $y_1 = \operatorname{cosec}^{-1}\frac{x}{a}$,

$$\frac{dy_1}{dx} = -\frac{a}{x\sqrt{x^2 - a^2}}.$$

13. Let $y = \text{versin}^{-1}\dfrac{x}{a}$;

$$\therefore \quad x = a \text{ versin } y = a - a\cos y,$$

$$\frac{dx}{dy} = a\sin y;$$

$$\therefore \frac{dy}{dx} = \frac{1}{a\sin y};$$

but $\cos y = \dfrac{a-x}{a}$;

$$\therefore \frac{dy}{dx} = \frac{1}{a\sqrt{\left(1 - \frac{a^2 - 2ax + x^2}{a^2}\right)}},$$

$$= \frac{1}{\sqrt{(2ax - x^2)}}.$$

And if $y_1 = \text{covers } x$,

$$\frac{dy_1}{dx} = -\frac{1}{\sqrt{(2ax - x^2)}}.$$

14. **To find the Differential Coefficient of a Function of a Function.**

Let $y = f(u)$, where u is a function of x. Let $u = F(x)$. It is required to find the differential coefficient of y with respect to x. Let x become $x+h$, and, in consequence, let u become $u+m$, and y, $y+k$.

Then $\quad y+k = f(u+m)$,

$$k = f(u+m) - f(u);$$

$$\therefore \frac{k}{h} = \frac{f(u+m) - f(u)}{m} \times \frac{m}{h}.$$

Now the limiting value of $\dfrac{f(u+m) - f(u)}{m}$ as m becomes small, is the differential coefficient of $f(u)$ with respect to u. This has been denoted by $f'(u)$. The limiting value of $\dfrac{m}{h}$, as h

becomes small is $\frac{du}{dx}$, and of $\frac{k}{h}$ is $\frac{dy}{dx}$. Therefore, when h becomes small,
$$\frac{dy}{dx}=f'(u)\times\frac{du}{dx}.$$

For example. It has been shown that if $y=e^x$, then $\frac{dy}{dx}=e^x$. But let it be required to find the differential coefficient of $e^{x\sin x}$.

$$\text{Let } x\sin x=u.$$
$$\text{Then if we put } y=e^u=f(u),$$
$$f'(u)=e^u, \text{ also } \frac{du}{dx}=\sin x+x\cos x;$$
$$\therefore\ \frac{dy}{dx}=e^u\times(\sin x+x\cos x),$$
$$=e^{x\sin x}(\sin x+x\cos x).$$

An expression is said to be differentiated when its differential coefficient is found.

EXAMPLES:

(1) $y=\log\{x+(x^2+a^2)^{\frac{1}{2}}\}$. $\qquad \frac{dy}{dx}=\frac{1}{\sqrt{a^2+x^2}}$.

(2) $y=\log\frac{1+x+x^2}{1-x+x^2}$. $\qquad \frac{dy}{dx}=\frac{2(1-x^2)}{1+x^2+x^4}$.

In this case the work of differentiation is simplified if y is put into the form $\log(1+x+x^2)-\log(1-x+x^2)$, and each term differentiated.

(3) $y=\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$. $\qquad \frac{dy}{dx}=1+\frac{x}{\sqrt{x^2-1}}$.

In all cases the expression to be differentiated should be simplified, as far as possible, before the differentiation is

performed. Here, for example, we obtain by multiplying numerator and denominator by $\sqrt{x+1}+\sqrt{x-1}$,

$$y=x+\sqrt{x^2-1},$$

and the result is easily obtained.

(4) $y=\dfrac{1+\tan x}{1-\tan x}$. $\qquad \dfrac{dy}{dx}=\sec^2\left(\dfrac{\pi}{4}+x\right)$.

Here $y=\tan\left(\dfrac{\pi}{4}+x\right)$.

(5) $$y=x^x.$$

Then $$\log y=x\log x.$$

Differentiate both sides of this equation with respect to x. Now $\log y$ is a function of x;

$$\therefore \quad \frac{d}{dx}(\log y)=\frac{d}{dy}(\log y)\times\frac{dy}{dx};$$

$$\therefore \quad \frac{1}{y}\cdot\frac{dy}{dx}=1+\log x;$$

$$\therefore \quad \frac{dy}{dx}=x^x(1+\log x).$$

This artifice of finding the differential coefficient of $\log y$ often shortens the process.

Find $\dfrac{dy}{dx}$ in each of the following examples:

(6) $y=e^{x^x}$. $\qquad \dfrac{dy}{dx}=e^{x^x}x^x(1+\log x)$.

(7) $y=\log\tan x$. $\qquad$ Ans. $2\operatorname{cosec} 2x$.

(8) $y=\tan^{-1}\dfrac{4\sqrt{x}}{1-4x}$. $\qquad$ " $\dfrac{2}{(1+4x)\sqrt{x}}$.

(9) $y=\sqrt{\left(\dfrac{1+x}{1-x}\right)}$. $\qquad$ " $\dfrac{1}{(1-x)\sqrt{(1-x^2)}}$.

(10) $y=x^{\cos x}$. $\qquad$ " $x^{\cos x}\left(\dfrac{\cos x}{x}-\sin x.\log x\right)$.

(11) $y=a^{\frac{1}{\sin^{-1}x}}$. Ans. $-\dfrac{\log_e a}{\sqrt{(1-x^2)}(\sin^{-1}x)^2}\,a^{\frac{1}{\sin^{-1}x}}$.

(12) $y=\sin^{-1}\sqrt{\sin x}$. „ $\frac{1}{2}\sqrt{(1+\operatorname{cosec} x)}$.

(13) $y=\tan^{-1}\sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)}$.

$$\text{Here } \tan^2 y=\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\dfrac{x}{2}}{2\cos^2\dfrac{x}{2}},$$

$$=\tan^2\frac{x}{2};$$

$$\therefore\ y=\frac{x}{2} \text{ and } \frac{dy}{dx}=\tfrac{1}{2}.$$

(14) $y=\cos^{-1}\dfrac{x^2-1}{x^2+1}$. Ans. $-\dfrac{2}{1+x^2}$.

(15) $y=\log\dfrac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}$. „ $\dfrac{2\sqrt{2}(1-x^2)}{1+x^4}$.

(16) $y=\log\tan\dfrac{x}{2}$. „ $\operatorname{cosec} x$.

15. **The Differentiation of Implicit Functions.**

If y is given as a function of x, the value of $\dfrac{dy}{dx}$ can be obtained by the preceding methods. If $f(x, y)=0$ is an equation connecting y and x, y is said to be an **Implicit Function** of x, and if the equation cannot be solved so as to obtain y in terms of x, the following method for finding $\dfrac{dy}{dx}$ must be used.

Let x become $x+h$, and in consequence let y become $y+k$.

Then $f(x+h, y+k)=0$.

Now $\frac{f(x+h,\ y+k)-f(x+h,\ y)}{k}\times\frac{k}{h}+\frac{f(x+h,\ y)-f(x,\ y)}{h}=0$ identically. Therefore the limit of this expression when h becomes small $=0$. The limit of $\frac{k}{h}$ is $\frac{dy}{dx}$ by definition.

The limit of $\frac{f(x+h,\ y+k)-f(x+h,\ y)}{k}$ is found by the process which would be adopted if x were constant and y alone varied.

This limit is denoted by $\left(\frac{df}{dy}\right)$.

Similarly the limit of $\frac{f(x+h,\ y)-f(x,\ y)}{h}$ is found by the method which would be adopted if x alone varied. Call it $\left(\frac{df}{dx}\right)$.

Thus $\left(\frac{df}{dy}\right)\frac{dy}{dx}+\left(\frac{df}{dx}\right)=0.$

For example:

Let $f(x,\ y)=x^3+y^3-3axy=0.$

Then $\left(\frac{df}{dx}\right)=3x^2-3ay,$

$\left(\frac{df}{dy}\right)=3y^2-3ax;$

$\therefore\ (y^2-ax)\frac{dy}{dx}+(x^2-ay)=0;$

$\therefore\quad \frac{dy}{dx}=-\frac{x^2-ay}{y^2-ax}.$

A similar method enables us to find the differential coefficient of a function of x and y with respect to x, y being a function of x.

If $u=f(x, y)$,

Let x, y, and u become $x+h$, $y+k$, and $u+l$ respectively.

Then

$$\frac{l}{h}=\frac{f(x+h,\ y+k)-f(x,\ y)}{h},$$

$$=\frac{f(x+h,\ y+k)-f(x+h,\ y)}{k}\cdot\frac{k}{h}+\frac{f(x+h,\ y)-f(x,\ y)}{h};$$

$\therefore$ when h becomes small, we have as before,

$$\frac{du}{dx}=\left(\frac{df}{dy}\right)\frac{dy}{dx}+\left(\frac{df}{dx}\right).$$

Examples :

(1) $x=y\log xy$, $\qquad \dfrac{dy}{dx}=\dfrac{x-y}{x(1+\log xy)}$.

(2) $x^y=y^x$, $\qquad \dfrac{dy}{dx}=\dfrac{y}{x}\cdot\dfrac{y-x\log y}{x-y\log x}$.

(3) $mx=\sin xy$, $\qquad \dfrac{dy}{dx}=\dfrac{m}{x\cos xy}-\dfrac{y}{x}$.

(4) $x^4+ax^2y-by^3=0$, $\dfrac{dy}{dx}=\dfrac{4x^3+2axy}{3by^2-ax^2}$.

16. If y and x are given as functions of a third variable z, y may be found as a function of x by eliminating z between the given equations, and the differential coefficient of y with respect to x found as before. If the elimination of z cannot be performed, the following method must be adopted :—

Let $y=f(z)$, and let $x=F(z)$.

Let z become $z+l$, and in consequence let x and y become $x+h$ and $y+k$ respectively.

$$\text{Then } x+h=F(z+l),$$

$$y+k=f(z+l);$$

$$\therefore\ \frac{k}{h}=\frac{\dfrac{f(z+l)-f(z)}{l}}{\dfrac{F(z+l)-F(z)}{l}}.$$

Therefore the limits of these expressions are equal when l, h, and k become small.

$$\therefore\ \frac{dy}{dx}=\frac{f'(z)}{F'(z)}.$$

For example:

$$\text{Find } \frac{dy}{dx} \text{ where } y=e^z \text{ and } x=\cos z.$$

$$\text{Here } f(z)=e^z \text{ and } f'(z)=e^z,$$

$$F(z)=\cos z \text{ and } F'(z)=-\sin z;$$

$$\therefore\ \frac{dy}{dx}=-\frac{e^z}{\sin z}=-\frac{y}{\sqrt{1-x^2}}.$$

This result could also be obtained by putting y equal to $e^{\cos^{-1}x}$.

Let $y=\sin t+\log\cos t$, $x=t+e^t$,

$$\frac{dy}{dx}=\frac{\cos t-\tan t}{1+e^t}.$$

Let $y=5z+z^5$, $x=\log\sqrt{\dfrac{1+z}{1-z}}+\tan^{-1}z$,

$$\frac{dy}{dx}=\frac{5}{2}(1-z^8).$$

17. The following differential coefficients should be committed to memory.

$$\frac{dx^n}{dx}=nx^{n-1}; \quad \frac{d\log_a x}{dx}=\frac{1}{x\log_e a}; \quad \frac{da^x}{dx}=a^x\log_e a.$$

$$\frac{d\sin x}{dx}=\cos x; \qquad \frac{d\cos x}{dx}=-\sin x.$$

$$\frac{d\tan x}{dx}=\sec^2 x; \qquad \frac{d\cot x}{dx}=-\operatorname{cosec}^2 x$$

$$\frac{d\sec x}{dx}=\frac{\sin x}{\cos^2 x}; \qquad \frac{d\operatorname{cosec} x}{dx}=-\frac{\cos x}{\sin^2 x}.$$

$$\frac{d\sin^{-1}x}{dx}=\frac{1}{\sqrt{1-x^2}}; \qquad \frac{d\cos^{-1}x}{dx}=-\frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d\tan^{-1}x}{dx}=\frac{1}{1+x^2}; \qquad \frac{d\cot^{-1}x}{dx}=-\frac{1}{1+x^2}.$$

$$\frac{d\sec^{-1}x}{dx}=\frac{1}{x\sqrt{x^2-1}}; \qquad \frac{d\operatorname{cosec}^{-1}x}{dx}=-\frac{1}{x\sqrt{x^2-1}}.$$

$$\frac{d\operatorname{vers}^{-1}x}{dx}=\frac{1}{\sqrt{2x-x^2}}; \qquad \frac{d\operatorname{covers}^{-1}x}{dx}=-\frac{1}{\sqrt{2x-x^2}}.$$

CHAPTER III.

SUCCESSIVE DIFFERENTIATION.

1. LET y be a function of x, then the differential coefficient of y with respect to x will in general be a function of x.

If $y=f(x)$, this function is denoted by $f'(x)$ or $\frac{dy}{dx}$. The differential coefficient of this quantity can then be found by the methods given in the last chapter, and the value so obtained is called the second differential coefficient of y with respect to x. It is written $f''(x)$ or $\frac{d^2y}{dx^2}$.

Thus if $y=\log_e x$,

$$\frac{dy}{dx}=\frac{1}{x}, \text{ and}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}.$$

2. Similarly the third differential coefficient is the differential coefficient of the second, and in general the n^{th} differential coefficient is the differential coefficient of the $(n-1)^{\text{th}}$.

This is denoted by $f^n(x)$ or $\frac{d^ny}{dx^n}$.

3. The value of $\frac{d^ny}{dx^n}$ for any assigned value of n is in general

found by successive applications of the methods of differentiation, but in some cases a general expression can be found.

For example:

(1) If $y=a^x$,

$$\frac{dy}{dx}=a^x \log_e a;$$

$$\therefore \frac{d^2y}{dx^2}=\log_e a\frac{da^x}{dx}=(\log_e a)^2 a^x,$$

and $$\frac{d^n y}{dx^n}=(\log_e a)^n a^x.$$

For if this holds for any number n,

$$\frac{d^{n+1}y}{dx^{n+1}}=(\log_e a)^n \frac{da^x}{dx}=(\log_e a)^{n+1}a^x;$$

but this is true when $n=2$, and therefore universally.

(2) If $y=x^m$,

Then $$\frac{dy}{dx}=mx^{m-1},$$

$$\frac{d^2y}{dx^2}=m(m-1)x^{m-2}.$$

If r is not greater than m,

$$\frac{d^r y}{dx^r}=m(m-1) \ . \ . \ . \ (m-r+1)x^{m-r}.$$

For if $\frac{d^{r-1}y}{dx^{r-1}}=m(m-1) \ . \ . \ . \ (m-r+2)x^{m-r+1}$, we get the above value by differentiating $\frac{d^{r-1}y}{dx^{r-1}}$ with respect to x; but this is true when $r=2$, and therefore universally.

If r is greater than m, and m is a positive integer,

$$\frac{d^r y}{dx^r}=0.$$

$$\text{Also } \frac{d^m x^m}{dx^m}=\underline{|m}.$$

(3) Let $y=\sin x$.

$$\text{Then} \quad \frac{dy}{dx}=\cos x=\sin\left(x+\frac{\pi}{2}\right),$$

$$\frac{d^2 y}{dx^2}=\cos\left(x+\frac{\pi}{2}\right)=\sin\left(x+2\ \frac{\pi}{2}\right),$$

$$\text{And generally } \frac{d^n y}{dx^n}=\sin\left(x+n\frac{\pi}{2}\right),$$

For differentiating again, we get

$$\frac{d^{n+1}y}{dx^{n+1}}=\cos\left(x+n\frac{\pi}{2}\right)=\sin\left(x+\overline{n+1}\frac{\pi}{2}\right),$$

which is true when $n=1$, and therefore universally.

4. Leibnitz's Theorem. To find the n^{th} Differential Coefficient of the product of two functions of x.

Let u and v be functions of x. Then

$$\frac{d^n uv}{dx^n}=u\frac{d^n v}{dx^n}+n\frac{du}{dx}\,\frac{d^{n-1}v}{dx^{n-1}}+\ldots+\frac{n(n-1)\ldots(n-r+2)}{\underline{|r-1}}\times$$

$$\frac{d^{r-1}u}{dx^{r-1}}\cdot\frac{d^{n-r+1}v}{dx^{n-r+1}}+\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}\cdot\frac{d^r u}{dx^r}\cdot\frac{d^{n-r}v}{dx^{n-r}}+\ldots+\frac{d^n u}{dx^n}\cdot v.$$

For assuming this expansion, differentiate both sides of this equation with respect to x; we obtain—

$$\frac{d^{n+1}uv}{dx^{n+1}}=u\frac{d^{n+1}v}{dx^{n+1}}+\frac{du}{dx}\cdot\frac{d^nv}{dx^n}+n\frac{du}{dx}\cdot\frac{d^nv}{dx^n}+\ \dots$$

$$+\frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}}\cdot\frac{d^ru}{dx^r}\cdot\frac{d^{n-r+1}v}{dx^{n-r+1}}+\frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\times$$

$$\frac{d^ru}{dx^r}\cdot\frac{d^{n-r+1}v}{dx^{n-r+1}}+\ \dots\ +\frac{d^{n+1}u}{dx^{n+1}}\cdot v.$$

That is, collecting similar terms,

$$\frac{d^{n+1}uv}{dx^{n+1}}=u\frac{d^{n+1}v}{dx^{n+1}}+(n+1)\frac{du}{dx}\cdot\frac{d^nv}{dx^n}+\ \dots\ +\frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}}\times$$

$$\left\{1+\frac{n-r+1}{r}\right\}\cdot\frac{d^ru}{dx^r}\cdot\frac{d^{n-r+1}v}{dx^{n-r+1}}+\ \dots\ +\frac{d^{n+1}u}{dx^{n+1}}\cdot v,$$

$$=u\frac{d^{n+1}v}{dx^{n+1}}+\ \dots\ +\frac{(n+1)n\dots(\overline{n+1}-r+1)}{\underline{|r}}\cdot\frac{d^ru}{dx^r}\cdot\frac{d^{n+1-r}v}{dx^{n+1-r}}$$

$$+\ \dots\ +\frac{d^{n+1}u}{dx^{n+1}}\cdot v.$$

Now this is of the same form as the assumed expansion for $\frac{d^nuv}{dx^n}$, with $n+1$ written in place of n. But the expansion holds when $n=1$, because $\frac{duv}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$, and therefore it holds universally.

5. This theorem is useful when one of the functions is such that its n^{th} differential coefficient can be easily expressed, and the other such that its n^{th} differential coefficient vanishes for a small value of n.

For example:

Let $u=x^3$, and $v=\log_e x$.

Now $\frac{dv}{dx}=\frac{1}{x}$, $\frac{d^2v}{dx^2}=-x^{-2}$, $\frac{d^3v}{dx^3}=\underline{|2}\,.\,x^{-3}$,

and generally $\frac{d^nv}{dx^n}=(-1)^{n-1}\underline{|n-1}\,x^{-n}$;

$$\frac{du}{dx}=3x^2;\ \frac{d^2u}{dx^2}=6x;\ \frac{d^3u}{dx^3}=6,$$

and each of the higher differential coefficients vanishes.

Therefore

$$\frac{d^nx^3\log x}{dx^n}=x^3\cdot(-1)^{n-1}\underline{|n-1}\,x^{-n}+n\cdot 3x^2(-1)^{n-2}\underline{|n-2}\cdot x^{-n+1}$$

$$+\frac{n(n-1)}{\underline{|2}}\cdot 6x(-1)^{n-3}\underline{|n-3}\cdot x^{-n+2}$$

$$+\frac{n(n-1)(n-2)}{\underline{|3}}\cdot 6(-1)^{n-4}\underline{|n-4}\cdot x^{-n+3},$$

$$=(-1)^{n-1}\cdot x^{3-n}\cdot\underline{|n}\left\{\frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}\right\},$$

$$=(-1)^n\cdot 6\underline{|n-4}\cdot x^{3-n}.$$

This assumes that n is not less than 4. For smaller values of n, the successive differential coefficients are most easily found by actual differentiation.

$$\text{Thus }\frac{d^2x^3\log x}{dx^2}=x(6\log x+5),$$

$$\frac{d^3x^3\log x}{dx^3}=6\log x+11.$$

6. The expansion also enables us in many cases to find a relation between consecutive differential coefficients of an expression.

For example:

$$\text{If } y=\cos\log x+\sin\log x,$$

$$\text{prove that } x^2\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2+1)\frac{d^ny}{dx^n}=0.$$

Such relations are most easily proved by proving the particular case in which $n=0$, and applying Leibnitz's Theorem to find the n^{th} differential coefficient of the expression so found.

$$\text{Thus } \frac{dy}{dx}=\frac{-\sin\log x+\cos\log x}{x},$$

$$\frac{d^2y}{dx^2}=\frac{-(\sin\log x+\cos\log x)+(\sin\log x-\cos\log x)}{x^2},$$

$$\therefore\ x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

Now, taking the n^{th} differential coefficient of each side of this equation by Leibnitz's Theorem, we get

$$x^2\frac{d^{n+2}y}{dx^{n+2}}+n\cdot 2x\frac{d^{n+1}y}{dx^{n+1}}+\frac{n(n-1)}{\underline{|2}}\cdot 2\frac{d^ny}{dx^n}+x\frac{d^{n+1}y}{dx^{n+1}}+n\frac{d^ny}{dx^n}+\frac{d^ny}{dx^n}=0,$$

which reduces to the required relation.

Leibnitz's Theorem is often expressed shortly by the following notation.

Let D^ru represent $\frac{d^ru}{dx^r}$, and D'^rv, $\frac{d^rv}{dx^r}$.

Then if $(D+D')^nuv$ be expanded by the binomial theorem, regarding D and D' as algebraical quantities and $\frac{d^ru}{dx^r}\cdot\frac{d^sv}{dx^s}$ be written for $D^rD'^suv$ in each term, we obtain the expansion of $\frac{d^nuv}{dx^n}$. Thus with this notation we may write

$$\frac{d^nuv}{dx^n}=(D+D')^nuv.$$

EXAMPLES:

(1) $y=x^2\log x,\qquad \dfrac{d^2y}{dx^2}=3+2\log x.$

(2) $y=e^{-x}\cos x,\qquad \dfrac{d^4y}{dx^4}=-4e^{-x}\cos x.$

(3) $y=(x^2+a^2)\tan^{-1}\dfrac{x}{a},\quad \dfrac{d^3y}{dx^3}=\dfrac{4a^3}{(x^2+a^2)^2}.$

(4) If $y=\{x+\sqrt{(x^2-1)}\}^m$, show that

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2-m^2)\frac{d^ny}{dx^n}=0.$$

(5) $y=\dfrac{1}{x^2-a^2},\quad \dfrac{d^ny}{dx^n}=\dfrac{(-1)^n\lfloor n}{2a}\left\{\dfrac{(x+a)^{n+1}-(x-a)^{n+1}}{(x^2-a^2)^{n+1}}\right\}.$

Put y into the form $\dfrac{1}{2a}\left\{\dfrac{1}{x-a}-\dfrac{1}{x+a}\right\}.$

(6) $y=\dfrac{6x^3+5x^2-7}{3x^2-2x-1},$

$$\frac{d^ny}{dx^n}=(-1)^n\lfloor n\left\{\frac{5\cdot 3^n}{(3x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right\}$$

if n is greater than 2.

(See Hall and Knight's *Higher Algebra*, chap. xxiii. § 318.)

CHAPTER IV.

TAYLOR'S THEOREM.

1. Axiom:

If $f(x)$ is finite and continuous for all values of x between the values $x=a$, and $x=b$, and if $f(a)=0$, and $f(b)=0$, then $f'(x)$ must equal zero for some value of x between a and b.

This follows from Chap. I. § 6. For $f'(x)$ must change from positive to negative, or from negative to positive as x changes from a to b.

2. Let

$$\frac{\underline{|n+1}}{x^{n+1}}\left\{f(a+x)-f(a)-xf'(a)-\frac{x^2}{\underline{|2}}f''(a)-\ .\ .\ .\ -\frac{x^n}{\underline{|n}}f^n(a)\right\}=X,$$

where a is a constant, and n any positive integer.

Then the expression

$$f(a+h)-f(a)-hf'(a)-\frac{h^2}{\underline{|2}}f''(a)-\ .\ .\ .\ -\frac{h^n}{\underline{|n}}f^n(a)-\frac{h^{n+1}}{\underline{|n+1}}X,$$

where h is independent of x,

is equal to zero when $h=0$, and when $h=x$.

Therefore the Differential Coefficient of this quantity equals zero for some value of h between 0 and x.

Let that value be αx, where α is a proper fraction. That is—

$$f'(a+h)-f'(a)-hf''(a)-\ .\ .\ .\ -\frac{h^{n-1}}{\underline{|n-1}}f^n(a)-\frac{h^n}{\underline{|n}}X$$

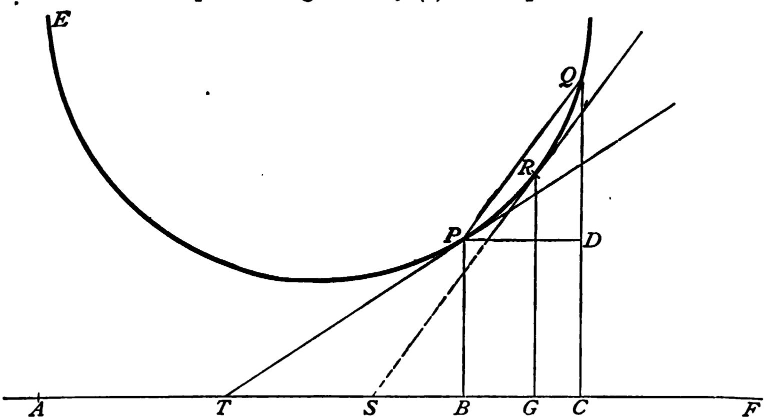
equals zero when $h=\alpha x$; but it also equals zero when $h=0$, therefore its Differential Coefficient is zero for some value of h between 0 and αx. Let that value be βx.

By continuing this process to $n+1$ differentiations, we find that
$$f^{n+1}(a+h)-X$$
must equal zero for some value of h between 0 and μx, where μ is a proper fraction. Let this value be θx. Then substituting this value of h,
$$f^{n+1}(a+\theta x)-X=0.$$

Substitute this value of X in the first equation, multiply by $\dfrac{x^{n+1}}{\lfloor n+1}$, and transpose the negative terms.

Thus
$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{\lfloor 2}f''(a)+\ .\ .\ .\ +\frac{x^n}{\lfloor n}f^n(a)+\frac{x^{n+1}}{\lfloor n+1}f^{n+1}(a+\theta x).$$
This expansion is called **Taylor's Theorem.**

3. Taking two terms of this expansion, a geometrical meaning may be assigned to it, depending upon the geometrical interpretation given to $f'(x)$ in Chapter I.



Using the figure and notation of Chapter I. § 8, draw a tangent to the circle, making the tangent parallel to PQ. Now it is possible to draw this at a point between P and Q: let that point be R, and RS the tangent. Draw RG perpendicular to AF.

Now the value of tan RSF is $f'(AG)$ by Chap. I. § 8;

$$\therefore \frac{QD}{PD}=f'(AG);$$

$$\therefore \frac{QC-PB}{BC}=f'(AB+BG);$$

$$\therefore \frac{f(x+h)-f(x)}{h}=f'(x+\theta h);$$

where θ is a proper fraction.

4. Let $a=0$; then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\underline{|2}}f''(0)+\ \ldots\ +\frac{x^n}{\underline{|n}}f^n(0)+\frac{x^{n+1}}{\underline{|n+1}}f^{n+1}(\theta x).$$

This is known as **Maclaurin's Expansion.**

The last term in either series is called the **remainder** after $n+1$ terms.

5. The value of the fraction θ cannot in general be found, but in many cases it is unnecessary to find it. If x is less than unity and $f^{n+1}(\theta x)$ is finite, $f(x)$ can be made to differ from the sum of the first $n+1$ terms by a quantity which can be made as small as we please by taking n large enough—that is, by taking a sufficient number of terms.

For example, let $f(x)=\sin x$;

Then $f(0)=\sin 0=0$;

$f'(x)=\cos x,\ f'(0)=\cos 0=1$;

$f''(x)=-\sin x,\ f''(0)=0$;

$f'''(x)=-\cos x,\ f'''(0)=-1$;

$f^n(x)=\sin\left(x+n\frac{\pi}{2}\right),\ f^n(0)=\sin n\frac{\pi}{2}.$

Therefore, substituting in Maclaurin's Series,

$$\sin x=x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\ \ldots\ +\frac{x^{n+1}}{\underline{|n+1}}\sin\left(\theta x+\overline{n+1}\frac{\pi}{2}\right).$$

Now the sine of any angle is less than unity; therefore the remainder may be made numerically smaller than any quantity we please, if x is a proper fraction. Then, subject to this condition, we may assume

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots$$

where the sum of the series differs from $\sin x$ by a quantity which may be made as small as we please by taking a sufficient number of terms.

6. If the remainder after $n+1$ terms is not required, and if it may be assumed that the given function can be expanded in ascending powers of the variable of which it is a function, the following method will determine the coefficients of the terms.

Let $f(a+x) = A_0 + A_1x + A_2x^2 + \ldots A_nx^n + \ldots$ (1)

where A_0, $A_1 \ldots A_n$ are independent of x.

Let $x=0$, $\therefore f(a) = A_0$.

Differentiate both sides of equation (1),

$$f'(a+x) = A_1 + 2A_2x + \ldots + nA_nx^{n-1} + \ldots$$

Let $x=0$, $\therefore f'(a) = A_1$, and so on.

Differentiate n times, then

$$f^n(a+x) = \underline{|n}.A_n + \text{terms in } x,\ x^2, \text{ etc.}$$

Let $x=0$, $\therefore f^n(a) = \underline{|n}.A_n$,

and by substituting for A_n we obtain the same result as before.

7. Many expansions can be found by employing this method and the method of Undetermined Coefficients. For example,

let it be required to expand $\sin^{-1}x$ in ascending powers of x.

Let $\sin^{-1}x = A_0 + A_1x + A_2x^2 + \dots + A_nx^n + \dots$ (1)

If $x=0$, $\sin^{-1}x=0$, $\therefore A_0=0$.

Differentiate both sides of equation (1). Then

$$\frac{1}{\sqrt{1-x^2}} = A_1 + 2A_2x + \dots + nA_nx^{n-1} + \dots$$

Now expand $(1-x^2)^{-\frac{1}{2}}$ by the Binomial Theorem:

$$(1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\cdot x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{\underline{|2}}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{\underline{|3}}(-x^2)^3 + \dots$$

$$= 1 + \tfrac{1}{2}x^2 + \frac{1.3}{\underline{|2}}\left(\frac{x^2}{2}\right)^2 + \frac{1.3.5}{\underline{|3}}\left(\frac{x^2}{2}\right)^3 + \dots$$

Thus equating coefficients of similar powers of x, we obtain

$$A_1 = 1,\ 2A_2 = 0,\ 3A_3 = \tfrac{1}{2},$$

and generally $nA_n = 0$ if n is an even number, and

$$= \frac{1.3.5.\dots(n-2)}{1.2.3.\dots\times\frac{n-1}{2}} \cdot \frac{1}{2^{\frac{n-1}{2}}} \text{ if } n \text{ is odd.}$$

Now $\left(1.2.3.\dots\times\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}} = 2.4.6.\dots(n-1)$.

Therefore

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \dots$$

EXAMPLES:

(1) $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$

(2) $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$

8. By a similar method the expansion of $\sin x$ can be obtained.

Let $\sin x = A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n + \ldots$

Now by putting $x=0$, we find $A_0=0$.

Differentiate,

$$\cos x = A_1 + 2A_2 x + 3A_3 x^2 + \ldots + nA_n x^{n-1} + \ldots$$

Let $x=0$, $\therefore A_1 = 1$. Differentiate again,

$$-\sin x = 1.2A_2 + 2.3A_3 x + 3.4A_4 x^2 + \ldots + (n-1)nA_n x^{n-2} + \ldots$$
$$= -A_0 - \quad A_1 x - \quad A_2 x^2 - \ldots - \quad A_{n-2} x^{n-2} - \ldots$$

$$\therefore \text{ in general, } (n-1)nA_n = -A_{n-2}.$$

Now $A_0 = 0$; $\therefore A_2 = 0 = A_4 = A_6 \ldots$

$$A_1 = 1; \quad \therefore A_3 = -\frac{1}{\underline{|3}}, \quad A_5 = -\frac{A_3}{4.5} = \frac{1}{\underline{|5}}, \text{ etc.}$$

$$\therefore \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots$$

9. In some simple cases the value of the proper fraction θ, which occurs in the expansion of $f(x+h)$ may be found. For example, let only two terms of the expansion be taken, then

$$f(x+h) = f(x) + hf'(x+\theta h) \quad . \quad . \quad (1)$$

Now if $f(x) = x^3$, $f'(x) = 3x^2$, and equation (1) is equivalent to

$$(x+h)^3 = x^3 + h3(x+\theta h)^2.$$

But $(x+h)^3 = x^3 + h(3x^2 + 3xh + h^2)$;

$\therefore$ comparing these values

$$6x\theta h + 3\theta^2 h^2 = 3xh + h^2,$$

a quadratic equation from which θ can be found.

One of the values so found is a proper fraction. For

$$\theta=\frac{-x+\sqrt{x^2+xh+\frac{1}{3}h^2}}{h}.$$

And $\sqrt{x^2+xh+\frac{1}{3}h^2}-x<h$,

because $x^2+xh+\frac{1}{3}h^2<x^2+2xh+h^2$.

EXAMPLES :

(1) $\log_e(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ \ldots$

(2) $\log\cos x=-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\ \ldots$

(3) $e^{x\sin x}=1+x^2+\frac{x^4}{3}+\frac{x^6}{\cancel{720}}+\ \ldots$ 5

(4) $e^{\tan^{-1}x}=1+x+\frac{x^2}{2}-\frac{x^3}{6}+\ \ldots$

CHAPTER V.

MAXIMA AND MINIMA VALUES OF A FUNCTION OF ONE VARIABLE.

1. LET $f(x)$ be a continuous function of x; then if $f(a)$ is greater than $f(a+h)$ and greater than $f(a-h)$ when h is taken indefinitely small, then $f(a)$ is called **a Maximum Value** of the function. If $f(a)$ is less than $f(a+h)$ and less than $f(a-h)$, $f(a)$ is called **a Minimum Value.**

2. By Taylor's theorem,

$$\frac{f(a+h)-f(a)}{h}=f'(a)+\alpha,$$

$$\text{and } \frac{f(a-h)-f(a)}{h}=-f'(a)+\beta,$$

where α and β become small, as h approaches zero.

Therefore if $f'(a)$ is positive, $f(a+h)$ is greater than $f(a)$ and $f(a-h)$ is less than $f(a)$, h being supposed positive, and $f(a)$ is neither a maximum nor a minimum value of the function. The same holds if $f'(a)$ is negative. Therefore there can be no maximum or minimum value unless $f'(a)=0$.

Let $f'(a)=0$, then, by Taylor's Theorem,

$$\frac{f(a+h)-f(a)}{\frac{1}{2}h^2}=f''(a)+\alpha',$$

$$\frac{f(a-h)-f(a)}{\frac{1}{2}h^2}=f''(a)+\beta',$$

where α' and β' approach zero along with h. Therefore, if

$f''(a)$ is positive, $f(a+h)$ is greater than $f(a)$, and $f(a-h)$ is greater than $f(a)$ when h becomes indefinitely small. Then $f(a)$ is a minimum value of the function. If $f''(a)$ is negative, $f(a+h)$ and $f(a-h)$ are less than $f(a)$, then $f(a)$ is a maximum value of the function.

3. Thus, to find the maxima and the minima values of a function $f(x)$, find all the values of x which satisfy the equation $f'(x)=0$, if $a_1, a_2 \ldots$ are these values, then $f(a_1), f(a_2) \ldots$ are maxima or minima values of the given function according as $f''(a_1), f''(a_2) \ldots$ are negative or positive.

For example, if $f(x)=2x^3+9x^2-60x+75$,

$$f'(x)=6x^2+18x-60.$$

The values $x=2$ or -5 make $f'(x)=0$.

Also $f''(x)=6(2x+3)$.

Now if $x=2$, $f''(x)$ is positive, and if $x=-5$, $f''(x)$ is negative;

$\therefore f(2)$ is a minimum value, 7,

and $f(-5)$ is a maximum value, 350.

4. The value of x, which has been found from the equation $f'(x)=0$, might also make $f''(x)$, $f'''(x)$, etc., each equal to zero. Let it make the first n differential coefficients zero, where n is an even number. Then

$$\frac{f(a+h)-f(a)}{\frac{1}{\lfloor n+1}h^{n+1}}=f^{n+1}(a)+a_1,$$

and
$$\frac{f(a-h)-f(a)}{\frac{1}{\lfloor n+1}h^{n+1}}=-f^{n+1}(a)+\beta_1,$$

where a_1 and β_1 become small when h is small.

Therefore, there will be neither maximum nor minimum

value of $f(x)$ for the value a of x, unless $f^{n+1}(a)$ is also zero. If this condition holds, we have, since $n+2$ is an even number,

$$\frac{f(a+h)-f(a)}{\frac{1}{\lfloor n+2}h^{n+2}}=f^{n+2}(a)+\alpha'_1,$$

$$\frac{f(a-h)-f(a)}{\frac{1}{\lfloor n+2}h^{n+2}}=f^{n+2}(a)+\beta'_1,$$

where α'_1 and β'_1 diminish with h.

Thus if $f^{n+2}(a)$ is positive, $f(a)$ is a minimum value of the function, and if $f^{n+2}(a)$ is negative, $f(a)$ is a maximum.

EXAMPLES :

Find the maxima and minima values of the following expressions :

(1) $3x^4-8x^3-42x^2-48x+608.$

Minimum$=0$; show the value is neither when $x=-1$.

(2) $(x-1)^4(x+2)^3.$

Maximum $=\frac{9^3\times 12^4}{7^7}$, minimum$=0$.

(3) $\frac{a^2}{x}+\frac{b^2}{a-x}.$

Maximum $=\frac{(a-b)^2}{a}$, minimum $=\frac{(a+b)^2}{a}$.

(4) $x^{\frac{1}{x}}.$ $e^{\frac{1}{e}}=$maximum.

(5) $\sin 2x+2\sin x.$ $\frac{3\sqrt{3}}{2}=$maximum.

(6) $\frac{x}{1+x\tan x}.$ $\tan\left(\frac{\pi}{4}-\frac{a}{2}\right)=$maximum, where $a=\cdot 739\ldots$ a root of the equation $\cos x=x$.

(7) $a^3 \sin^3 \frac{\pi x}{a}$.

Calling this expression $f(x)$, we can express it in terms of $\sin \frac{3\pi x}{a}$.

$$f(x) = \frac{a^3}{4}\left(3 \sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a}\right);$$

$$\therefore f'(x) = \frac{3\pi a^2}{4}\left(\cos \frac{\pi x}{a} - \cos \frac{3\pi x}{a}\right).$$

$$\text{If } f'(x) = 0, \cos \frac{\pi x}{a} = \cos \frac{3\pi x}{a};$$

$$\therefore \frac{3\pi x}{a} = 2n\pi \pm \frac{\pi x}{a},$$

where n is any integer; that is

$$x = na, \text{ or } \frac{n}{2}a.$$

The maxima values are given when x equals $2ma + \frac{a}{2}$, m being any integer, the minima values, when $x = 2ma - \frac{a}{2}$; when $x = na$, the values of $f(x)$ are neither maxima nor minima, because $f'''(na) = \frac{3\pi^3}{4}\left(9 \cos 3n\pi - \cos n\pi\right)$, which is not zero, while $f''(na) = \frac{3\pi^2 a}{4}\left(3 \sin 3n\pi - \sin n\pi\right)$, that is, zero.

(8) $(x-a)^2(x-b)^3$, where a is not equal to b.

The value $x = a$, gives a maximum if $a < b$, and a minimum if $a > b$.

The value $x = \frac{2a+3b}{5}$ gives a minimum and a maximum in the same cases. If $x = b$, the value is neither.

5. In applying this method to geometrical problems, the quantity whose maximum value is to be found is expressed

in terms of any variable on which its value depends, and the expression differentiated with respect to this variable.

Examples:

(1) Find the greatest isosceles triangle that can be inscribed in a given circle. Let d be the diameter of the given circle, and 2θ the vertical angle of the triangle. Taking this angle as the unknown variable on which the area depends, it can be shown that the area $= d^2 \sin\theta \cos^2\theta$.

Thus the greatest area is $\frac{2\sqrt{3}}{9} \cdot d^2$.

(2) Find the isosceles triangle of greatest perimeter that can be inscribed in a circle.

An equilateral triangle.

(3) Find the smallest isosceles triangle that can be described about a given circle.

An equilateral triangle.

(4) The isosceles triangle of least perimeter described about a given circle is equilateral.

(5) Find the volume of the greatest cylinder that can be inscribed in a sphere of radius r.

$$\frac{4\sqrt{3}}{9}\pi r^3.$$

(6) Find the volume of the greatest right circular cone that can be inscribed in a sphere of radius r.

$$\frac{32}{81}\pi r^3.$$

(7) Find the convex surface of a right circular cone inscribed in a given sphere, when that surface is a maximum.

$$\frac{8\sqrt{3}}{9}\pi r^2.$$

CHAPTER VI.

EVALUATION OF INDETERMINATE EXPRESSIONS.

1. If a function of x is such that when a definite or an infinite value is assigned to x, no definite value can be assigned to the function, the function is called **Indeterminate** for that value of x.

For example, let $f(x)$ and $F(x)$ be two functions of x such that $f(a)=0$, and $F(a)=0$; then no value can be assigned to the expression $\dfrac{f(a)}{F(a)}$.

The limiting value of such an expression, as x approaches the assigned value, is called the value of the function for the particular value of x.

2. **To find the Value of a Function which assumes the form $\frac{0}{0}$.**

Let $f(x)$ and $F(x)$ be defined as above.

Then $f(a+h)=f(a)+hf'(a+\theta h)$, by Taylor's theorem,

and $F(a+h)=F(a)+hF'(a+\theta' h)$.

But $f(a)=0=F(a)$ by hypothesis;

$$\therefore \frac{f(a+h)}{F(a+h)}=\frac{f'(a+\theta h)}{F'(a+\theta' h)}.$$

Now as h becomes small $f'(a+\theta h)$ and $F'(a+\theta' h)$ approach

the values $f'(a)$ and $F'(a)$, therefore the limiting value of $\frac{f(a+h)}{F(a+h)}$ as h becomes small $=\frac{f'(a)}{F'(a)}$.

If this expression should again be of the form $\frac{0}{0}$, similar reasoning shows that the limit is $\frac{f''(a)}{F''(a)}$, or in general $\frac{f^n(a)}{F^n(a)}$, where the n^{th} is the first differentiation in which both functions do not vanish.

If the first $n-1$ differential coefficients of $f(a)$ and of $F(a)$ equal zero, and $f^n(a)=0$ while $F^n(a)$ is not zero, then the limiting value is zero; if $F^n(a)=0$ and $f^n(a)$ is not zero, the value of $\frac{f(x)}{F(x)}$ increases indefinitely as x approaches a.

For example: If $x=0$, the expression

$$\frac{e^x+e^{-x}+2\cos x-4}{x^4}$$

takes the form $\frac{0}{0}$. By the rule the limit is the same as the value of

$$\text{Lt}\ \frac{e^x-e^{-x}-2\sin x}{4x^3} \text{ when } x \text{ approaches } 0.$$

And this is again of the same form.

$$\therefore \text{Limit} = \text{Lt}\ \frac{e^x+e^{-x}-2\cos x}{12x^2}, \text{ when } x=0.$$

$$= \text{Lt}\ \frac{e^x-e^{-x}+2\sin x}{24x} \quad . \quad . \quad . \quad .$$

$$= \text{Lt}\ \frac{e^x+e^{-x}+2\cos x}{24} \quad . \quad . \quad . \quad .$$

$$= \frac{1}{6}.$$

All other indeterminate forms can be reduced to the preceding. The others are $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0.

3. If $f(a)=\infty$, and $F(a)=\infty$, $\frac{f(a)}{F(a)}$ can be put into the form $\frac{\frac{1}{F(a)}}{\frac{1}{f(a)}}$, which takes the form $\frac{0}{0}$, and can be evaluated by the preceding method.

Let L be the limit of $\frac{f(a)}{F(a)}$, then

$$\mathrm{L}=\frac{-\frac{1}{\{F(a)\}^2}F'(a)}{-\frac{1}{\{f(a)\}^2}f'(a)}, \text{ by the preceding rule,}$$

$$=\left\{\frac{f(a)}{F(a)}\right\}^2\frac{F'(a)}{f'(a)},$$

$$=\mathrm{L}^2\frac{F'(a)}{f'(a)},$$

$$\therefore \mathrm{L}=\frac{f'(a)}{F'(a)}.$$

That is, the limiting value is found by the same rule as before.

This proof assumes that L is not 0. If the limit of $\frac{f(a)}{F(a)}$ is 0, the limit of $\frac{f'(a)}{F'(a)}$ is also 0. For the limit of $\frac{F(a)+f(a)}{F(a)}$ is 1, and therefore the limit of $\frac{F'(a)+f'(a)}{F'(a)}$ is also 1. That is, the limit of $\frac{f'(a)}{F'(a)}$ is 0.

4. If $f(a)=0$, and $F(a)=\infty$, then since

$$f(a)\times F(a)=\frac{f(a)}{\frac{1}{F(a)}},$$

this form can be evaluated by the same method.

5. If $f(a)=\infty$, and $F(a)=\infty$, $f(a)-F(a)$ is indeterminate. Writing the expression

$$\frac{1-\frac{F(a)}{f(a)}}{\frac{1}{f(a)}},$$

it takes the form $\frac{0}{0}$, if the limiting value of $\frac{F(a)}{f(a)}$ is unity. If the limit of $\frac{F(a)}{f(a)}$ is not unity, the given expression is equal to a fraction whose numerator is not zero, and whose denominator becomes indefinitely small; that is, $f(a)-F(a)$ becomes numerically indefinitely great.

6. The remaining forms are reduced to the first by finding the limiting value of the logarithm of the given quantity. Thus, if

$$\mathrm{L}=\{F(a)\}^{f(a)},$$

$$\log \mathrm{L}=f(a)\log\{F(a)\},$$

and in each case this is of the form $0\times\infty$.

EXAMPLES:

(1) The limit of $x \log x$ when $x=0$, is 0.

This is of the form $0 \times \infty$.

Now
$$\text{Lt}\,\frac{\log x}{\dfrac{1}{x}} = \text{Lt}\,\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}},$$
$$=\text{Lt}\,(-x)=0.$$

(2) Find the value of $(\cos x)^{\frac{1}{x}}$ when $x=0$.

This is of the form 1^∞.

Let $y=(\cos x)^{\frac{1}{x}}$, $\therefore$ $\log_e y = \dfrac{\log \cos x}{x}$, which is of the form $\dfrac{0}{0}$;
$$\therefore \text{Lt}\,\log_e y = \text{Lt}\,\frac{\log \cos x}{x},$$
$$=\text{Lt}\,\frac{-\tan x}{1}=0;$$
$$\therefore \text{Lt}\,y=e^0=1.$$

(3) Find the value of $(\cos x)^{\frac{1}{x^2}}$ when $x=0$. Ans. $\dfrac{1}{\sqrt{e}}$.

(4) $(1-x)\tan x\dfrac{\pi}{2}$, when $x=1$. . . Ans. $\dfrac{2}{\pi}$.

(5) $\dfrac{x\log(1+x)}{1-\cos x}$, when $x=0$. . . . . Ans. 2.

(6) $\dfrac{1}{x}-\cot x$, when $x=0$.

This is of the form $\infty-\infty$, but the given expression may be written $\dfrac{\tan x - x}{x \tan x}$, which is of the form $\dfrac{0}{0}$. The value approaches 0.

(7) $\dfrac{a^x-b^x}{x}$, when $x=0$. . . . . . Ans. $\log_e\dfrac{a}{b}$.

(8) $\dfrac{x-\sin x}{x^3}$, when $x=0$. . . . Ans. $\dfrac{1}{6}$.

(9) $(\cos x)^{\cot x}$ when $x=0$. . . . Ans. 1.

(10) $x^{\frac{1}{x}}$ when $x=\infty$. . . . . Ans. 1.

(11) $x^{\frac{1}{1-x}}$ when $x=1$. . . . . . Ans. e^{-1}.

(12) $\dfrac{m\sin\theta-\sin m\theta}{\theta(\cos\theta-\cos m\theta)}$, when $\theta=0$. . . Ans. $\dfrac{m}{3}$.

This expression can be readily evaluated if the numerator and denominator be expanded by Maclaurin's Theorem.

Thus the fraction

$$=\frac{-\dfrac{\theta^3}{\underline{|3}}(m-m^3)+\dfrac{\theta^5}{\underline{|5}}(m-m^5)-\text{etc.}}{\theta\left\{-\dfrac{\theta^2}{\underline{|2}}(1-m^2)+\dfrac{\theta^4}{\underline{|4}}(1-m^4)-\text{etc.}\right\}},$$

$$=\frac{\dfrac{m}{6}(1-m^2)-\text{terms in }\theta^2}{\dfrac{1}{2}(1-m^2)-\text{terms in }\theta^2},$$

$$=\frac{m}{3}\text{ when }\theta\text{ becomes small.}$$

(13) $\dfrac{\tan nx-n\tan x}{n\sin x-\sin nx}$, when $x=0$. . . Ans. 2.

CHAPTER VII.

INTEGRATION.

1. LET $f(x)$ be a function of x, which is continuous and finite between the values $x=a$ and $x=b$ inclusive, then the limiting value of the sum of the series

$$h\{f(a)+f(a+h)+f(a+2h)+\ldots+f(b)\}$$

when h is taken indefinitely small, and the number of terms in the series is $\frac{b-a}{h}+1$, is called the **Definite Integral** of $f(x)$ between the limits a and b, and it is written $\int_a^b f(x)dx$.

2. In many cases this limiting value can be found by ordinary algebraical methods. For example, to find the value of $\int_0^1 x^2 dx$, that is the limiting value of the sum of the series

$$h\{0+h^2+(2h)^2+\ldots+(nh)^2\},$$

where $nh=1$, and h is taken indefinitely small.

$$\begin{aligned} \text{The sum} \quad &= h^3\{1^2+2^2+3^2+\ldots+n^2\}, \\ &= h^3\left\{\frac{n(n+1)(2n+1)}{6}\right\}, \\ &= \frac{nh(nh+h)(2nh+h)}{6}, \\ &= \frac{1(1+h)(2+h)}{6}, \\ &= \frac{1}{3} \text{ in the limit, when } h \text{ becomes very small.} \end{aligned}$$

3. If the number of terms in the series is $\frac{b-a}{h}$, the limit of the sum is still $\int_a^b f(x)dx$, because the limit of the last term, that is of $hf(b)$, is 0, when h becomes small. The same is true if a finite number of terms is omitted.

4. Let $\phi(x)$ be another function of x; then, since

$$h\{f(a)+\ldots f(b)\}+h\{\phi(a)+\ldots\phi(b)\}=h\{[f(a)+\phi(a)]+\ldots [f(b)+\phi(b)]\},$$

$$\int_a^b f(x)dx+\int_a^b \phi(x)dx=\int_a^b\{f(x)+\phi(x)\}dx.$$

Again, if c be any constant,

$$h\{cf(a)+cf(a+h)+\ldots cf(b)\}=ch\{f(a)+f(a+h)+\ldots+f(b)\};$$

$$\therefore \int_a^b cf(x)dx=c\int_a^b f(x)dx.$$

5. **To find the Sum of the Series by the method of Integration.**

Let $F(x)$ be a function of x, such that

$$\frac{dF(x)}{dx}=f(x);$$

then if h be a small increment in the value of x, the limiting value of

$$\frac{F(x+h)-F(x)}{h}=f(x), \text{ when } h \text{ is taken very small.}$$

Therefore, in general,

$$F(x+h)-F(x)=hf(x)+kh,$$

where k is a quantity which becomes indefinitely small, when h becomes small. Calling k_1, k_2 . . . the values of this

quantity when a, $a+h$ . . . are written successively for x, we have

$$F(a+h)-F(a)=hf(a)+k_1h,$$
$$F(a+2h)-F(a+h)=hf(a+h)+k_2h,$$
$$\text{etc.} \quad = \quad \text{etc.}$$
$$F(b+h)-F(b)=hf(b)+k_nh, \text{ where } b=a+(n-1)h.$$

Therefore, adding, we obtain

$$F(b+h)-F(a)=h\{f(a)+f(a+h)+ \ . \ . \ . \ +f(b)\}$$
$$+h\{k_1+k_2+ \ . \ . \ . \ k_n\}.$$

Let k_r be the greatest of the quantities k_1, k_2 . . . k_n and k_s the least, then the sum of the series

$$h\{f(a)+f(a+h)+ \ . \ . \ . \ +f(b)\}$$

lies between the values of

$$F(b+h)-F(a)-nhk_r, \text{ and of}$$
$$F(b+h)-F(a)-nhk_s.$$

But when h becomes small, k_r and k_s also become small, while nh remains finite, being equal to $b-a+h$, therefore the sum of the series approaches $F(b)-F(a)$.

Thus to find the required limit, find the function $F(x)$, of which $f(x)$ is the Differential Coefficient, substitute first b, then a, for x in $F(x)$, and subtract the latter value from the former.

The quantity $F(x)$ is called the **Indefinite Integral** of $f(x)$ with respect to x, and it is written

$$F(x)=\int f(x)dx.$$

For example:

$$\frac{d(\frac{1}{3}x^3)}{dx}=x^2,$$
$$\therefore \quad \tfrac{1}{3}x^3=\int x^2dx.$$
$$\therefore \ \tfrac{1}{3}(b^3-a^3)=\int_a^b x^2dx.$$

$\int f(x)dx$ is often called shortly the integral of $f(x)$.

The process of finding $F(x)$ from $f(x)$ is called **Integration,** and the function $f(x)$ is said to be **integrated.**

If C is a quantity independent of x, $\frac{dC}{dx}=0$. Then since by definition $\frac{dF(x)}{dx}=f(x)$, we have

$$\frac{d}{dx}\{F(x)+C\}=f(x).$$

Therefore the integral of $f(x)$ is $F(x)+C$, C being any constant. This constant can be determined only from the data of the special problem. It obviously disappears from the definite integral, for

$$\int_a^b f(x)dx=\{F(b)+C\}-\{F(a)+C\},$$
$$=F(b)-F(a).$$

If two Functions are equal to each other, their Integrals can differ only by a Constant.

Let $F(x)$ be the integral of $f(x)$, and $F_1(x)$ of $f_1(x)$, and let $f(x)=f_1(x)$.

$$\text{Then}\quad \frac{dF(x)}{dx}=f(x)=\frac{dF_1(x)}{dx};$$
$$\therefore \frac{d\{F(x)-F_1(x)\}}{dx}=0,$$
$$\therefore F(x)-F_1(x)=\text{a constant.}$$

It is evident that $\int_a^b f(x)dx=\int_a^b f(y)dy$,

for neither of these expressions contains x or y.

6. The Definite Integral $\int_a^b f(x)dx$ has been defined as the limit of the sum of the series

$$h\{f(a)+f(a+h)+\ .\ .\ .\quad f(a+\overline{n-1}h)\},$$

when h is small.

Let A be the greatest of the quantities $f(a), f(a+h) \ldots f(b)$, and B the least, then

$$\int_a^b f(x)dx < h.nA, \text{ and} > h.nB. \quad \text{That is,}$$

$$< (b-a+h)A, \text{ and} > (b-a+h)B,$$

when h is indefinitely small;

$$\therefore \int_a^b f(x)dx < (b-a)A, \text{ and} > (b-a)B;$$

$\therefore$ There is some value C between A and B, such that

$$\int_a^b f(x)dx = (b-a)C.$$

Now, since $f(x)$ is continuous between the values $x=a$, and $x=b$, $f(x)$ must equal C for some value of x between a and b. Say $x=c$;

$$\therefore C=f(c), \text{ and}$$

$$\int_a^b f(x)dx = (b-a)f(c).$$

7. The following indefinite integrals are known from the results in the Differential Calculus:—

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad \int \sin x dx = -\cos x + C,$$

$$\int \frac{1}{x}.dx = \log_e x + C, \qquad \int \cos x dx = \sin x + C,$$

$$\int a^x dx = \frac{a^x}{\log_e a} + C, \qquad \int \sec^2 x dx = \tan x + C,$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + C,$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C, \qquad \text{or} = -\cos^{-1}\frac{x}{a} + C,$$

$$\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C, \quad \text{or} = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + C,$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad \text{or} = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C.$$

8. The constant C has not in general the same value in these integrals, and in some cases a relation between the constants may be found. Thus

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C=-\cos^{-1}x+C'.$$

Now, whatever be the value of x, if the least value of the angles be taken,

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2};$$

$$\therefore\ C'-C=\frac{\pi}{2}.$$

9. If a function to be integrated is not of a known form, it may often be made to depend on one of the known integrals by the following methods:—

First, The method of **Substitution.**

Let $f(x)$ be the function to be integrated. Let y be any function of x, then if for x its value in terms of y is substituted, let $f(x)$ become $\phi(y)$.

The differential coefficient with respect to x of the indefinite integral $\int f(x)dx$ is, by definition, $f(x)$. The differential coefficient of $\int\phi(y)\cdot\frac{dx}{dy}\cdot dy$ with respect to y is $\phi(y)\frac{dx}{dy}$, and therefore with respect to x is $\left\{\phi(y)\frac{dx}{dy}\right\}\frac{dy}{dx}$; that is $\phi(y)$ or $f(x)$.

Therefore $\int f(x)dx$ and $\int\phi(y)\frac{dx}{dy}dy$ can differ only by a constant.

$$\therefore\quad \int f(x)dx=\int\phi(y)\frac{dx}{dy}dy+C.$$

EXAMPLES:

(1) Find $\int \frac{dx}{\sqrt{(c^2+2ax-x^2)}}$.

Putting this in the form $\int \frac{dx}{\sqrt{\{c^2+a^2-(a-x)^2\}}}$ we see that it reduces to a known integral if $a-x=y$.

$$\text{Then} \quad \frac{dx}{dy}=-1;$$

$$\therefore \int \frac{dx}{\sqrt{(c^2+2ax-x^2)}}=\int \frac{-dy}{\sqrt{\{(c^2+a^2)-y^2\}}},$$

$$=\cos^{-1}\frac{y}{\sqrt{a^2+c^2}}+C,$$

$$=\cos^{-1}\frac{a-x}{\sqrt{a^2+c^2}}+C.$$

(2) $\int \tan x dx=\int \frac{\sin x}{\cos x}dx.$

Now let $\cos x=y$;

$$\therefore x=\cos^{-1}y; \quad \therefore \frac{dx}{dy}=\frac{-1}{\sqrt{1-y^2}};$$

$$\therefore \int \frac{\sin x}{\cos x}dx=-\int \frac{\sqrt{1-y^2}}{y}\times\frac{1}{\sqrt{1-y^2}}dy,$$

$$=-\int \frac{dy}{y}=-\log y+C,$$

$$=C-\log \cos x.$$

In this example the substitution could be effected more easily thus:

$$\cos x=y; \quad \therefore \frac{dx}{dy}=-\frac{1}{\sin x};$$

$$\therefore \int \frac{\sin x}{\cos x}dx=-\int \frac{\sin x}{y}.\frac{1}{\sin x}dy=-\int \frac{dy}{y}.$$

In general, when the function to be integrated is a fraction,

the numerator of which is the differential coefficient of the denominator, by putting the denominator equal to y, the integral reduces to the form $\int \frac{dy}{y}$, that is, $\log y$.

$$\text{Thus } \int \frac{2x+a}{c+ax+x^2}\,dx = \log(c+ax+x^2)+C.$$

$$\int \cot x\,dx = \log \sin x + C.$$

$$\int \frac{1+\cos x}{x+\sin x}\,dx = \log(x+\sin x)+C.$$

(3) Find $\int \frac{1}{e^x+e^{-x}}\,dx.$

$$\int \frac{1}{e^x+\frac{1}{e^x}}\,dx = \int \frac{e^x}{1+e^{2x}}\,dx.$$

Now let $e^x = y$, $\therefore \frac{dx}{dy} = \frac{1}{y}$;

$$\therefore \int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{y}{1+y^2} \times \frac{1}{y}\,dy,$$

$$= \int \frac{dy}{1+y^2} = \tan^{-1} y + C,$$

$$= \tan^{-1} e^x + C.$$

Products and powers of sines and cosines should be reduced to single functions before being integrated. Thus:

$$\int \sin^3 x\,dx = \int \frac{1}{4}(3\sin x - \sin 3x)\,dx,$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C.$$

(4) Find $\int \frac{dx}{\sqrt{a^2+x^2}}$.

$$\text{Let } y=x+\sqrt{(a^2+x^2)};$$

$$\therefore \frac{dy}{dx}=1+\frac{x}{\sqrt{(a^2+x^2)}},$$

$$=1+\frac{x}{y-x}=\frac{y}{y-x}.$$

$$\text{Thus } \int \frac{dx}{\sqrt{(a^2+x^2)}}=\int \frac{1}{y-x}\times\frac{y-x}{y}\,dy=\int \frac{dy}{y},$$

$$=\log y=\log\{x+\sqrt{(a^2+x^2)}\}+C.$$

10. *Secondly*, The Method of **Integration by Parts.**

Let the function to be integrated be the product of two quantities, u and v, one of which, say u, can be integrated. Let w be $\int u dx$ so that $\frac{dw}{dx}=u$.

$$\text{Now } \frac{dwv}{dx}=vu+w\frac{dv}{dx};$$

$$\therefore \text{ integrating, } wv=\int vu dx+\int w\cdot\frac{dv}{dx}\cdot dx.$$

Thus if the expression $w\frac{dv}{dx}$ can be integrated, the required integral is found:

$$\int uv dx=wv-\int w\cdot\frac{dv}{dx}\cdot dx.$$

The factor u may be taken any factor of the given expression; for example, it may be taken equal to unity. Then, since $\int 1\cdot dx=x$,

$$\int v dx=xv-\int x\cdot\frac{dv}{dx}\cdot dx.$$

EXAMPLES:

(1) Find $\int xe^x dx$.

$$\text{Taking } u=e^x, \text{ we have } w=\int u dx=e^x;$$

$$\therefore \int xe^x dx = xe^x - \int e^x dx,$$

$$= e^x(x-1).$$

(In this and the following examples the constant is omitted for convenience.)

(2) $\int \log x \; dx$. Taking $u=1$, we get

$$\int \log x \; dx = x \log x - \int x \times \frac{1}{x} dx,$$

$$= x \log x - \; x.$$

In many cases a second or third application of the same method may be necessary.

For example:

(3) To find $\int x^3 a^x dx$.

Integrating by parts,

$$\int x^3 a^x dx = x^3 \frac{a^x}{\log_e a} - \int \frac{3}{\log_e a} x^2 a^x dx,$$

$$= \frac{x^3 a^x}{\log_e a} - \frac{3}{\log_e a} \left\{ x^2 \frac{a^x}{\log_e a} - \int \frac{2}{\log_e a} x a^x dx \right\},$$

$$= \frac{x^3 a^x}{\log_e a} - \frac{3x^2 a^x}{(\log_e a)^2} + \frac{6}{(\log_e a)^2} \left[x \frac{a^x}{\log_e a} - \int \frac{a^x}{\log_e a} dx \right],$$

$$= \frac{a^x}{\log_e a} \left\{ x^3 - \frac{3x^2}{\log_e a} + \frac{6x}{(\log_e a)^2} - \frac{6}{(\log_e a)^3} \right\}.$$

(4) $$\int e^x \sin 3x dx = e^x \sin 3x - 3\int e^x \cos 3x dx,$$

$$= e^x \sin 3x - 3\left(e^x \cos 3x + 3\int e^x \sin 3x dx\right);$$

transposing,

$$10\int e^x \sin 3x dx = e^x(\sin 3x - 3\cos 3x);$$

$$\therefore \int e^x \sin 3x dx = \frac{e^x}{10}(\sin 3x - 3\cos 3x).$$

11. The case in which u is put equal to 1, leads by successive integrations to **Bernoulli's Series.**

$$\int v dx = xv - \int x\frac{dv}{dx}dx,$$

$$= xv - \frac{x^2}{2}\cdot\frac{dv}{dx} + \int\frac{x^2}{2}\cdot\frac{d^2v}{dx^2}dx,$$

$$= xv - \frac{x^2}{2}\cdot\frac{dv}{dx} + \frac{x^3}{\underline{|3}}\cdot\frac{d^2v}{dx^2} - \int\frac{x^3}{\underline{|3}}\cdot\frac{d^3v}{dx^3}dx;$$

and in general,

$$\int v dx = xv - \frac{x^2}{\underline{|2}}\cdot\frac{dv}{dx} + \ \ldots\ -(-1)^n\frac{x^n}{\underline{|n}}\frac{d^{n-1}v}{dx^{n-1}} + (-1)^n\int\frac{x^n}{\underline{|n}}\cdot\frac{d^nv}{dx^n}dx.$$

12. Another expansion in ascending powers of x, with constant coefficients, can be obtained by Maclaurin's Theorem.

Let $\int v dx = y$, then $\frac{dy}{dx} = v$, $\frac{d^2y}{dx^2} = \frac{dv}{dx}$, etc. Then y_0, the value of y when $x=0$, will be constant because it does not contain x.

Call this constant c;

$$\therefore \int v dx = c + xv_0 + \frac{x^2}{\underline{|2}}\left(\frac{dv}{dx}\right)_0 + \ldots + \frac{x^n}{\underline{|n}}\left(\frac{d^{n-1}v}{dx^{n-1}}\right)_0 + \ldots,$$

where v_0, $\left(\frac{d^{n-1}v}{dx^{n-1}}\right)_0$ are the values of v, $\frac{d^{n-1}v}{dx^{n-1}}$ when $x=0$.

These expansions are seldom of practical value.

EXAMPLE:

$$\int(x^3+4x^2)dx = \frac{x^3}{\underline{|3}}\cdot 8 + \frac{x^4}{\underline{|4}}\cdot 6 = \frac{x^4}{4} + \frac{4}{3}x^3,$$

the integral being obtained by the process of differentiation only.

13. When the Indefinite Integral has been obtained, the Definite Integral is found from it by substituting first the upper limit, then the lower limit, and subtracting the latter result from the former. It is usually written thus:

$$\int_0^1 xe^x dx = \Big[e^x(x-1)\Big]_0^1 = 0-(-1) = 1;$$

$$\int_0^\infty \frac{dx}{1+x^2} = \Big[\tan^{-1}x.\Big]_0^\infty = \frac{\pi}{2}.$$

14. If $F(x) = \int f(x)dx$, we have, $\int_a^b f(x)dx = F(b) - F(a)$.

If c be any other value of x,

$$F(b)-F(a) = F(b)-F(c)+F(c)-F(a);$$

that is $\int_a^b f(x)dx = \int_c^b f(x)dx + \int_a^c f(x)dx$. . . (1)

Also $\int_a^b f(x)dx = -[F(a)-F(b)] = -\int_b^a f(x)dx$. . (2)

15. Again, when h becomes small,

$$\text{Lt}\, h\{f(0)+f(h)+\dots+f(b)\}\equiv \text{Lt}\, h\{f(b)+f(b-h)+\dots+f(0)\},$$

that is,
$$\int_0^b f(x)dx=\int_0^b f(b-x)dx, \qquad (3)$$

and the same limit also equals

$$\text{Lt}\, h\left\{f(0)+f(h)+\dots f\left(\frac{b}{2}\right)\right\}+\text{Lt}\, h\left\{f\left(\frac{b}{2}+h\right)+\dots+f(b)\right\},$$

that is,
$$\int_0^b f(x)dx=\int_0^{\frac{b}{2}} f(x)dx+\int_0^{\frac{b}{2}} f(b-x)dx,$$
$$=\int_0^{\frac{b}{2}}\{f(x)+f(b-x)\}dx. \qquad (4)$$

Therefore if $f(x)=f(b-x)$ from $x=0$ to $x=\frac{b}{2}$ inclusive,

$$\int_0^b f(x)\,dx=2\int_0^{\frac{b}{2}} f(x)dx. \qquad (5)$$

If $f(x)=-f(b-x)$ between the same limits,

$$\int_0^b f(x)dx=0. \qquad (6)$$

For example:

$$\int^{\pi}\cos^{2n}x dx=2\int_0^{\frac{\pi}{2}}\cos^{2n}x dx, \text{ and } \int_0^{\pi}\cos^{2n+1}x dx=0,$$

where n is any integer.

These formulae make it possible in some cases to find the definite integral even when the indefinite integral cannot be found.

For example:

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx, \text{ by (3)};$$

$$\therefore\ 2\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx.$$

The indefinite integral

$$\int\frac{\sin x dx}{1+\cos^2 x}=-\tan^{-1}\cos x.$$

[Put $\cos x=y$]

$$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx=-\frac{\pi}{2}\Big[\tan^{-1}\cos x\Big]_0^\pi,$$

$$=\frac{\pi^2}{4}.$$

EXAMPLES:

(1) $\displaystyle\int\frac{dx}{\sqrt{(1+x)}+(1+x)^{\frac{3}{2}}}=2\tan^{-1}\sqrt{(1+x)}.$

(2) $\displaystyle\int\frac{dx}{\sin x\cos x}=\log\tan x.$

(3) $\displaystyle\int\sin^2 x\cos^3 x dx=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x.$

(4) $\displaystyle\int\frac{x^2\tan^{-1}x}{1+x^2}dx=x\tan^{-1}x-\tfrac{1}{2}(\tan^{-1}x)^2-\tfrac{1}{2}\log(1+x^2).$

(5) $\displaystyle\int x^n\log x dx=\frac{x^{n+1}}{n+1}\left\{\log x-\frac{1}{n+1}\right\}.$

(6) $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}.$

This is found by putting

$$\frac{1}{x^2-a^2}=\frac{1}{2a}\left\{\frac{1}{x-a}-\frac{1}{x+a}\right\}.$$

(7) $\displaystyle\int\frac{dx}{a+bx+cx^2}=\frac{2}{\sqrt{(4ac-b^2)}}\cdot\tan^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}$

if $4ac>b^2$;

and $\displaystyle=\frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}$

if $4ac<b^2$.

(8) $\int \frac{1}{\cos x} dx \qquad = \log \sqrt{\frac{1+\sin x}{1-\sin x}}.$

(9) $\int \frac{1}{\sin x} dx \qquad = \log \tan \frac{x}{2}.$

(10) $\int \frac{2cx+d}{\sqrt{(cx^2+dx+e)}} dx = 2\sqrt{(cx^2+dx+e)}.$

(11) $\int \frac{bx+d}{\sqrt{(c^2+2ax-x^2)}} dx = (ab+d)\cos^{-1}\frac{a-x}{\sqrt{a^2+c^2}} - b\sqrt{(c^2+2ax-x^2)}.$

(12) $\int \sin^{-1} x dx \qquad = x \sin^{-1} x + \sqrt{(1-x^2)}.$

(13) $\int \tan^{-1} x dx \qquad = x \tan^{-1} x - \frac{1}{2}\log(1+x^2).$

(14) $\int (1-\cos x)^2 dx \qquad = \frac{3x}{2} - 2\sin x + \frac{1}{2}\sin x \cos x.$

— (15) $\int \frac{x+\sin x}{1+\cos x} dx \qquad = x \tan \frac{x}{2}.$

(16) $\int_1^2 \frac{dx}{x^{\frac{3}{2}}} \qquad = \frac{4-\sqrt{2}}{6}.$

(17) $\int_0^1 x \tan^{-1} x dx \qquad = \frac{\pi-2}{4}.$

(18) $\int_0^{\frac{\pi}{4}} \frac{dx}{\sin x + \cos x} \qquad = \frac{1}{\sqrt{2}} \log(1+\sqrt{2}).$

(19) $\int_0^{\frac{\pi}{2}} \log \sin x dx \qquad = \frac{\pi}{2} \log \frac{1}{2}.$ (Use § 15.)

(20) $\int_0^a \sqrt{\frac{a+x}{a-x}} dx \qquad = a\left(\frac{\pi}{2}-1\right).$

CHAPTER VIII.

FORMULAE OF REDUCTION.

1. THE integral $\int x^m(a+bx^n)^p dx$, where a, b, m, n, p are constants, cannot in general be found. If p is a positive integer, $(a+bx^n)^p$ may be expanded by the Binomial Theorem and each term integrated.

2. If $\frac{m+1}{n}$ is a positive integer, the value of the integral can be found.

For, let $p=\frac{r}{s}$, where r and s are integers.

Then putting $a+bx^n=y^s$, $x=\left(\frac{y^s-a}{b}\right)^{\frac{1}{n}}$;

$$\therefore \qquad \frac{dx}{dy}=\frac{1}{n}\cdot sy^{s-1}\, b^{-\frac{1}{n}}\left(y^s-a\right)^{\frac{1}{n}-1},$$

and the integral reduces to

$$\frac{b^{-\frac{m+1}{n}}s}{n}\int y^{r+s-1}(y^s-a)^{\frac{m+1-n}{n}}dy.$$

If therefore $\frac{m+1}{n}$ is a positive integer, the expression may be integrated as before.

Again, writing the integral in the form

$$\int x^m\{x^n(b+ax^{-n})\}^p dx \text{ or } \int x^{m+np}(b+ax^{-n})^p dx,$$

we see by applying the previous rule, that the integral can be found if

$$\frac{m+np+1}{-n} \text{ is a positive integer.}$$

3. When none of these conditions holds, the integral may be reduced to another integral which can in many cases be found. It may be reduced to any one of the following six forms:

$$\int x^{m+n}(a+bx^n)^{p-1}dx, \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

$$\int x^{m-n}(a+bx^n)^{p+1}dx, \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

$$\int x^m(a+bx^n)^{p-1}dx, \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

$$\int x^m(a+bx^n)^{p+1}dx, \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

$$\int x^{m+n}(a+bx^n)^p dx, \quad . \quad . \quad . \quad . \quad \text{(V.)}$$

$$\int x^{m-n}(a+bx^n)^p dx, \quad . \quad . \quad . \quad . \quad \text{(VI.)}$$

I. To obtain (I.), integrate the given expression by parts.

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}}{m+1}(a+bx^n)^p - \frac{bpn}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx.$$

II. To obtain (II.), integrate $x^{m-n}(a+bx^n)^{p+1}$ by parts.

$$\int x^{m-n}(a+bx^n)^{p+1}dx = \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \frac{b(p+1)n}{m-n+1}\int x^m(a+bx^n)^p dx.$$

To find the remaining forms, the given quantity must be written in the form $x^m(a+bx^n)^{p-1}(a+bx^n)$; or, expanding it,

$$ax^m(a+bx^n)^{p-1}+bx^{m+n}(a+bx^n)^{p-1},$$

and each term must be integrated. Thus—

III. Integrating by parts:

$$\int x^m(a+bx^n)^p dx=\frac{x^{m+1}}{m+1}(a+bx^n)^p-\frac{bpn}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx. \quad . \quad (1)$$

Also $$\int x^m(a+bx^n)^p dx=a\int x^m(a+bx^n)^{p-1}dx+b\int x^{m+n}(a+bx^n)^{p-1}dx. \quad . \quad (2)$$

Multiply (2) by $\frac{pn}{m+1}$, and add to (1);

$$\therefore \left(1+\frac{pn}{m+1}\right)\int x^m(a+bx^n)^p dx=\frac{x^{m+1}}{m+1}(a+bx^n)^p+\frac{apn}{m+1}\int x^m(a+bx^n)^{p-1}dx.$$

IV. Integrate by parts:

$$\int x^m(a+bx^n)^{p+1}dx=\frac{x^{m+1}}{m+1}(a+bx^n)^{p+1}-\frac{bn(p+1)}{m+1}\int x^{m+n}(a+bx^n)^p dx. \quad (1)$$

Also $$\int x^m(a+bx^n)^{p+1}dx=a\int x^m(a+bx^n)^p dx+b\int x^{m+n}(a+bx^n)^p dx. \quad . \quad (2)$$

Multiply (2) by $\frac{n(p+1)}{m+1}$, and add to (1);

$$\therefore \quad \left(1+\frac{\overline{p+1}n}{m+1}\right)\int x^m(a+bx^n)^{p+1}dx$$

$$=\frac{x^{m+1}}{m+1}(a+bx^n)^{p+1}+\frac{an(p+1)}{m+1}\int x^m(a+bx^n)^p dx,$$

which is the required form.

V. $\int x^m(a+bx^n)^{p+1}dx=\frac{x^{m+1}}{m+1}(a+bx^n)^{p+1}-\frac{bn(p+1)}{m+1}\int x^{m+n}(a+bx^n)^p dx. \quad (1)$

Also $\int x^m(a+bx^n)^{p+1}dx=a\int x^m(a+bx^n)^p dx+b\int x^{m+n}(a+bx^n)^p dx. \quad (2)$

Subtract (2) from (1) and we obtain

$$0=\frac{x^{m+1}}{m+1}(a+bx^n)^{p+1}-a\int x^m(a+bx^n)^p dx-\left(b+\frac{bn\overline{p+1}}{m+1}\right)\int x^{m+n}(a+bx^n)^p dx,$$

the required relation.

VI. $\int x^{m-n}(a+bx^n)^{p+1}dx$

$$=\frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1}-\frac{b(p+1)n}{m-n+1}\int x^m(a+bx^n)^p dx. \quad . \quad (1)$$

Also $\int x^{m-n}(a+bx^n)^{p+1}dx$

$$=a\int x^{m-n}(a+bx^n)^p dx+b\int x^m(a+bx^n)^p dx. \quad . \quad . \quad . \quad (2)$$

Subtract (2) from (1) and we get the required relation,

$$0=\frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1}-a\int x^{m-n}(a+bx^n)^p dx$$

$$-\left(b+\frac{bn\overline{p+1}}{m-n+1}\right)\int x^m(a+bx^n)^p dx.$$

4. These equations are called **Formulae of Reduction**. It will be seen that they enable the given integral to be reduced to another in which the index of the power of $(a+bx^n)^p$ is raised or depressed by unity, and the index of the other factor x^m is raised or depressed by n. But the process may be repeated on the integral so obtained, so that the index of

$(a+bx^n)^p$ may be raised or depressed by any integer, and that of the factor x^m by any integral multiple of n.

If it is required to reduce the integral to one in which $(a+bx^n)^{p-1}$ is a factor, then an expression of which $(a+bx^n)^p$ is a factor is integrated by parts (as in I. and III.), in all other cases $(a+bx^n)^{p+1}$.

If it is required to reduce the integral to one in which x^{m-n} is a factor, then an expression of which x^{m-n} is a factor is integrated by parts (as II. and VI.), in all other cases x^m.

Which formula is the most useful can generally be easily determined, but must be learned by experience.

5. Examples :

(1) Find the value of $\int\frac{dx}{(a^2+x^2)^2}$.

Here $m=0$, $n=2$, $p=-2$, and we see that an application of IV. will reduce this to a known integral.

Thus, integrating by parts,

$$\int(a^2+x^2)^{-1}dx=x(a^2+x^2)^{-1}+2\int x^2(a^2+x^2)^{-2}dx. \quad . \quad (1)$$

Also $$\int(a^2+x^2)^{-1}dx=a^2\int(a^2+x^2)^{-2}dx+\int x^2(a^2+x^2)^{-2}dx. \quad (2)$$

Multiply (2) by 2 and subtract (1) from the result:

$$\int(a^2+x^2)^{-1}\,dx=-x(a^2+x^2)^{-1}+2a^2\int(a^2+x^2)^{-2}\,dx;$$

$$\therefore \frac{1}{a}\tan^{-1}\frac{x}{a}=-x(a^2+x^2)^{-1}+2a^2\int(a^2+x^2)^{-2}\,dx;$$

$$\therefore \int\frac{dx}{(a^2+x^2)^2}=\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\tan^{-1}\frac{x}{a}.$$

(2) Find the value of $\int \sqrt{a^2-x^2}dx$.

This can be found by III.

Integrating by parts,

$$\int (a^2-x^2)^{\frac{1}{2}}dx = x(a^2-x^2)^{\frac{1}{2}} + \int x^2(a^2-x^2)^{-\frac{1}{2}}dx;$$

also $\int (a^2-x^2)^{\frac{1}{2}}dx = a^2\int (a^2-x^2)^{-\frac{1}{2}}dx - \int x^2(a^2-x^2)^{-\frac{1}{2}}dx;$

adding, we obtain,

$$2\int (a^2-x^2)^{\frac{1}{2}}dx = x(a^2-x^2)^{\frac{1}{2}} + a^2\int (a^2-x^2)^{-\frac{1}{2}}dx;$$

$$\therefore \int \sqrt{(a^2-x^2)}dx = \frac{x}{2}\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

(3) $\int \frac{x^m}{\sqrt{(a^2-x^2)}}dx$ can be reduced by means of equation VI. to the integral $\int \frac{x^{m-2}}{\sqrt{(a^2-x^2)}}dx$. If m is a positive integer, we shall obtain, by repeating the reduction, either the integral $\int \frac{x}{\sqrt{(a^2-x^2)}}dx$ or $\int \frac{dx}{\sqrt{(a^2-x^2)}}$, according as m is odd or even. And these are known integrals.

6. Two cases of failure may be noticed.

(1) If $m-n=-1$, the formulae II. and VI. cannot be used. But in this case the expression is immediately integrable, for

$$\int x^{n-1}(a+bx^n)^p dx = \frac{1}{bn(p+1)}(a+bx^n)^{p+1}.$$

(2) If $m=-1$, the remaining four formulae cannot be immediately applied. In this case let $x=\frac{1}{y}$. Then

$$\int x^{-1}(a+bx^n)^p dx = -\int y^{-pn-1}(b+ay^n)^p dy,$$

and all the formulae can be applied to this expression, if

p is not equal to -1. If p has this value, the integral is $-\dfrac{1}{na}\log(b+ay^n)$.

For example:

Find the value of $\displaystyle\int\frac{dx}{x\sqrt{(a^2-x^2)}}$.

Ans. $\dfrac{1}{a}\{\log x-\log(a+\sqrt{a^2-x^2})\}$.

7. The following is a useful case of reduction.

Since $\sin x=\dfrac{d(-\cos x)}{dx}$, we have

$$\int\sin^n x dx=-\cos x.\sin^{n-1}x+(n-1)\int\sin^{n-2}x\cos^2 x dx,$$

$$=-\cos x.\sin^{n-1}x+(n-1)\int\sin^{n-2}x(1-\sin^2 x)dx,$$

$$=-\cos x\sin^{n-1}x+(n-1)\int\sin^{n-2}x dx-(n-1)\int\sin^n x dx;$$

$$\therefore\ n\int\sin^n x dx=-\cos x\sin^{n-1}x+(n-1)\int\sin^{n-2}x dx.$$

Therefore, the integration of $\sin^n x$ is made to depend on that of $\sin^{n-2}x$; and if n is a positive integer, this will reduce finally to $\int\sin x\,dx$ or $\int dx$, that is, to $-\cos x$, or to x, according as n is an odd or an even number.

If the integration is taken between the limits $\frac{\pi}{2}$ and 0, since $\cos\frac{\pi}{2}=0$, and $\sin 0=0$, we have

$$\int_0^{\frac{\pi}{2}}\sin^n x dx=\frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x dx.$$

Similarly, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx = \frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx,$

and so on. Thus, if n is an odd integer,

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} \cdots \cdot\frac{2}{3};$$

because $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\, dx = 1.$ And if n is an even integer,

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2} \cdots \cdot\frac{1}{2}\cdot\frac{\pi}{2}.$$

Examples :

(1) $\displaystyle\int \frac{x^5}{\sqrt{(a^2-x^2)}}dx = -\frac{1}{15}\sqrt{(a^2-x^2)}(3x^4+4a^2x^2+8a^4).$

(2) $\displaystyle\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{3\sqrt{(1-x^2)}}\left(\frac{3-2x^2}{1-x^2}\right).$

(3) $\displaystyle\int_0^{2a} x\sqrt{(2ax-x^2)}\, dx = \pi\frac{a^3}{2}.$

(4) $\displaystyle\int_0^a \sqrt{(a^2-x^2)}\cos^{-1}\frac{x}{a}\, dx = \frac{a^2(4+\pi)}{16}.$

(5) $\displaystyle\int \sqrt{(x^2 \pm a^2)}\, dx = \frac{x}{2}\sqrt{(x^2 \pm a^2)} \pm \frac{a^2}{2}\log\{x+\sqrt{(x^2 \pm a^2)}\}.$

(6) $\displaystyle\int_0^a x(a^2-x^2)^{\frac{1}{2}}\, dx = \frac{a^3}{3}.$

(7) $\displaystyle\int \frac{(1-x^2)^{\frac{5}{2}}}{x}\, dx = \log\frac{x}{1+\sqrt{(1-x^2)}} + \frac{\sqrt{(1-x^2)}}{15}\{3x^4-11x^2+23\}.$

This integral may be reduced by the method given in § 6, but the integration is more easily performed if we put $x=\sin y$.

This substitution is often useful when $1-x^2$ occurs in the function to be integrated.

$$(8)\quad \int_1^{\frac{4}{5}} \frac{(1-x^2)^{\frac{3}{2}}}{x}\,dx = \frac{84}{125} - \log 2.$$

CHAPTER IX.

THE INTEGRATION OF RATIONAL FRACTIONS.

To find $$\int \frac{ax^n+bx^{n-1}+\ldots+k}{a_1x^m+b_1x^{m-1}+\ldots+k_1}dx,$$
where n and m are positive integers.

If n is not less than m, divide the numerator by the denominator. The expression to be integrated is thus reduced to a Rational Algebraical Function of x, and a fraction whose numerator is at most of the $(m-1)^{\text{th}}$ degree. Next, resolve this fraction into partial fractions (Hall and Knight's *Higher Algebra*, chap. xxiii.), and integrate each fraction separately.

The expression finally to be integrated is of the form—

$$\frac{Lx+M}{(ax^2+bx+c)^r}+\ldots+\frac{L'x+M'}{(ax^2+bx+c)}+\frac{A}{(x+d)^p}+\ldots+\frac{B}{(x+d)},$$

where L, M, L', M', A, B, etc., are constants, and r and p are positive integers.

The first integral can be reduced to an integrable form, (see Chap. viii. § 3, IV.).

$$\int\frac{A}{(x+d)^p}dx=\frac{A}{1-p}(x+d)^{-p+1} \text{ and } \int\frac{B}{(x+d)}=B\log(x+d),$$

$$\int\frac{L'x+M'}{ax^2+bx+c}dx=\frac{L'}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\left(M'-\frac{L'b}{2a}\right)\int\frac{dx}{ax^2+bx+c}.$$

The former integral is $\log(ax^2+bx+c)$; for the latter, see Chap. vii. ex. (7).

For example:

$$\int\frac{7+x}{(1+x)(1+x^2)}dx=\int\frac{3}{1+x}dx+\int\frac{4-3x}{1+x^2}dx,$$

(see Hall and Knight's *Algebra*, page 265.)

$$=3\log(1+x)-\int\left(\frac{3}{2}\cdot\frac{2x}{1+x^2}-\frac{4}{1+x^2}\right)dx.$$

$$=3\log(1+x)-\frac{3}{2}\log(1+x^2)+4\tan^{-1}x.$$

Examples:

(1) $$\int\frac{dx}{x^2(a^2+x^2)}=\frac{1}{a^2}\int\left(\frac{1}{x^2}-\frac{1}{a^2+x^2}\right)dx,$$

$$=-\frac{1}{a^2x}-\frac{1}{a^3}\tan^{-1}\frac{x}{a}.$$

(2) $$\int\frac{dx}{x(x-3)^2}=\int\left\{\frac{\frac{1}{9}}{x}+\frac{\frac{1}{3}}{(x-3)^2}-\frac{\frac{1}{9}}{(x-3)}\right\}dx,$$

$$=\frac{1}{9}\left\{\log\frac{x}{x-3}-\frac{3}{x-3}\right\}.$$

(3) $$\int\frac{dx}{(x-1)^2(x^2+4)}=\frac{1}{25}\int\left\{\frac{5}{(x-1)^2}-\frac{2}{(x-1)}+\frac{2x-3}{x^2+4}\right\}dx,$$

$$=\frac{1}{25}\left\{-\frac{5}{(x-1)}+\log\frac{x^2+4}{(x-1)^2}-\frac{3}{2}\tan^{-1}\frac{x}{2}\right\}.$$

(4) $$\int\frac{dx}{(x^2+1)(x^2+x+1)}=\frac{1}{2}\log\frac{x^2+x+1}{x^2+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

(5) $$\int\frac{x^2-1}{x^4+x^2+1}dx=\frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}.$$

(6) $\displaystyle\int\frac{1}{x^4+x^2+1}dx \quad =\frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1}$

$$+\frac{1}{2\sqrt{3}}\left\{\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1}\frac{2x-1}{\sqrt{3}}\right\}.$$

(7) $\displaystyle\int\frac{6x^3+5x^2-7}{3x^2-2x-1}dx \quad =x^2+3x+\log(3x+1)^{\frac{5}{3}}(x-1).$

(8) $\displaystyle\int\frac{7+x}{x^3+x^2+x+1}dx \quad =\log\left\{\frac{1+x}{\sqrt{(1+x^2)}}\right\}^3+4\tan^{-1}x.$

(9) $\displaystyle\int\frac{xe^x}{(e^x-1)^3}dx \quad =\frac{x}{2}-\frac{x}{2(e^x-1)^2}-\frac{1}{2(e^x-1)}-\frac{1}{2}\log(e^x-1).$

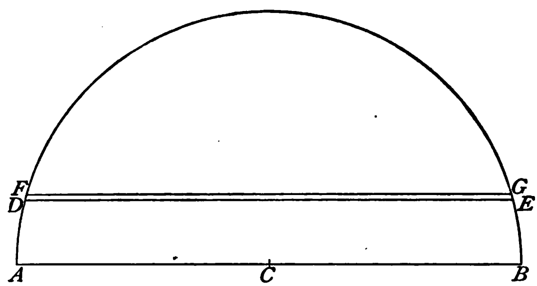
CHAPTER X.

SOME APPLICATIONS OF THE INTEGRAL CALCULUS.

THE following will serve as examples of applications which may be made of the Integral Calculus.

1. **To find the Area of a Circle of Radius** a.

Let AB be a diameter of the circle and C the centre. Draw DE a chord parallel to AB and at a distance x from it.



Then $DE=2\sqrt{(a^2-x^2)}$. (EUC. I. 47.)

Now if another chord FG be drawn parallel to DE, and at a distance h from it, the area of the part of the circle between DE and FG differs from

$$h\times 2\sqrt{(a^2-x^2)}$$

by a quantity which becomes smaller, the smaller h becomes. If then a number of chords be drawn parallel to AB, h being

the distance between two adjacent chords, the area of the semicircle differs from

$$h\{2a+2\sqrt{(a^2-h^2)}+2\sqrt{(a^2-2^2h^2)}+\ldots+2\sqrt{(a^2-a^2)}\},$$

by a quantity which is small when h is small. That is, the area of the semicircle

$$=2\int_0^a\sqrt{(a^2-x^2)}dx,$$

$$=\Big[x\sqrt{(a^2-x^2)}+a^2\sin^{-1}\frac{x}{a}\Big]_0^a,$$

$$=a^2\cdot\frac{\pi}{2}.$$

2. **To find the Volume of a Sphere of Radius a.**

If a plane be drawn through the centre, and the sphere divided into parts by planes drawn parallel to the plane through the centre, h being the thickness of each part, the method of the preceding example will give the volume of the hemisphere as the limiting value of the sum of the series

$$h\{\pi a^2+\pi(a^2-h^2)+\pi(a^2-2^2h^2)+\ldots+\pi(a^2-a^2)\},$$

when h becomes small. That is, the volume of the hemisphere

$$=\pi\int_0^a(a^2-x^2)dx,$$

$$=\pi\Big[a^2x-\frac{1}{3}x^3\Big]_0^a,$$

$$=\frac{2}{3}\pi a^3.$$

3. The Sum of a Series may often be deduced by the method of Integration from the known sum of another Series.

For example:

We have by division

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots \text{ to infinity, if } x \text{ is less than } 1.$$

Integrating both sides of this equation,

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots \text{ to infinity}+C,$$

where C is a constant.

To find its value, let $x=0$;

$$\therefore \quad \log 1=C;$$

$$\therefore \quad C=0.$$

4. If we expand $(1-x^2)^{-\frac{1}{2}}$ by the Binomial Theorem, we obtain the equation

$$\frac{1}{\sqrt{(1-x^2)}}=1+\frac{1}{2}\cdot x^2+\frac{1.3}{2.4}\cdot x^4+\frac{1.3.5}{2.4.6}\cdot x^6+\ldots.$$

Integrating both sides,

$$\sin^{-1}x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots.$$

The constant of integration is 0, because when $x=0$, $\sin^{-1}x=0$.

5. By division, we obtain,

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots\ldots \text{ to infinity,}$$

x being <1;

$$\therefore \tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\ldots,$$

the constant being 0, because $x=0$ when $\tan^{-1}x=0$.

6. The principal applications of the Calculus, namely, those to Dynamics and Physics, generally involve the methods of Analytical Geometry. For those applications special works on the subjects must be consulted.

MISCELLANEOUS EXAMPLES.

Find the differential coefficients of the following functions of x, from (1) to (22):—

(1) $\sqrt{\dfrac{2-x}{2+x}}$. (2) $\left\{\dfrac{x}{1-\sqrt{1-x^2}}\right\}^m$.

(3) $(a-x)^{3x^2}$. (4) $\log\sqrt[3]{\dfrac{a^3+a^2x+ax^2+x^3}{a^2+ax+x^2}}$.

(5) $\sqrt{\left\{a^2-b^2\cos^4\left(x-\dfrac{a}{2}\right)\right\}}$.

(6) $a^{3x}\log(1-x^2)$. (7) $\sin x\cos x\cos 2x\cos 4x$.

(8) $x^3\log\sqrt{\left(1-\sin\dfrac{a}{x}\right)\left(1-\cos\dfrac{a}{x}\right)}$.

(9) $\log(\sin 2x+\cos 2x)$.

(10) $\sin^{-1}\dfrac{3+4x}{5\sqrt{1+x^2}}$. (11) $\tan^{-1}\dfrac{bx-a}{ax+b}$.

(12) $\tan^{-1}\dfrac{1}{\cos x}$. (13) $\tan\cos^{-1}\dfrac{1}{x}$.

(14) $e^{ax}\sin^{-1}\{2x\sqrt{1-x^2}\}$.

(15) $\tan^{-1}\{x-\sqrt{1-x^2}\}$. (16) $\sec^{-1}\dfrac{a-x}{a-2x}$.

(17) $\log\tan^3\left(\dfrac{\pi}{4}+x\right)$. (18) $a^{3x^2}\cos^{-1}\dfrac{2x^2+1}{2x^2+3}$.

(19) $\tan^{-1}\left(\sin\dfrac{a}{x}+\cos\dfrac{a}{x}\right)$.

(20) $(\tan^{-1}x)^{1+x^2}$. (21) $\sin(\log x)^x$.

(22) $(\sin\log x)^x$.

(23) If $u=(\cos^{-1}ax)^2$, find $\dfrac{d^2u}{dx^2}$, and prove

$$(a^2x^2-1)\frac{d^{n+2}u}{dx^{n+2}}+(2n+1)a^2x\frac{d^{n+1}u}{dx^{n+1}}+n^2a^2\frac{d^nu}{dx^n}=0.$$

(24) If $y=\sin(m\tan^{-1}x)$, prove

$$(1+x^2)\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}+m^2y=0.$$

(25) If $y=\cos(m\sin^{-1}x)$, prove

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(m^2-n^2)\frac{d^ny}{dx^n}=0.$$

(26) If $y=e^{a\tan^{-1}x}$, prove

$$(1+x^2)\frac{d^{n+2}y}{dx^{n+2}}+(2nx+2x-a)\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)\frac{d^ny}{dx^n}=0.$$

(27) Find $\dfrac{dy}{dx}$ from the equation $\tan^{-1}xy=x\tan y$.

(28) If $\sin x+\log y=e^{x+y}$, find $\dfrac{dy}{dx}$.

(29) Expand $\log(2-x)$ by Maclaurin's Theorem as far as the term involving x^4.

(30) Expand $\dfrac{x}{\sin x}$ by Maclaurin's Theorem as far as the term involving x^4.

(31) Find the coefficient of x^7 in the expansion of $\log\{\sqrt{1+x^2}-x\}$ in ascending powers of x.

(32) Expand $\log(x-h)^{x-h}$ by Taylor's Theorem, in ascending powers of h, giving the remainder after the term involving h^4.

(33) Find the maxima and the minima values of

$$x^4-14x^2+24x+117.$$

(34) Find the maxima and the minima values of

$$9^x - 6x \log 3.$$

(35) Prove that $x \tan^{-1} x - \log \sqrt{1+x^2} - \frac{\pi x}{4}$ is never negative.

(36) Find the maxima and the minima values of

$$\cos x \cos (x+a) \cos (x-a).$$

(37) A right circular cone is circumscribed to a hemisphere, the base of the cone lying in the same plane as the base of the hemisphere. Find the form of the cone when (1) its convex surface, (2) its total surface, is a minimum.

(38) The semivertical angle of a right circular cone is a, and the height is h. Find the right circular cone (1) of greatest volume, (2) of greatest convex surface, that can be cut from it, the vertex of the inscribed cone coinciding with the centre of the base of the given cone.

(39) The lengths of the sides of a triangle are 4, 5, and 6 units respectively. Find the length of the shortest straight line that will bisect the triangle.

Find the limiting values of the following expressions from (40) to (43) :—

(40) $\dfrac{\log \sin 5x}{\log \sin 3x}$, when $x=0$. (41) $(x \log x)^x$, when $x=0$.

(42) $\dfrac{a^{2x} \sin mx - b^{2x} \sin nx}{\tan mx - \tan nx}$, when $x=0$.

(43) $\dfrac{\sin x - e^{\cos x}}{\log \tan \frac{x}{2} - \cos x}$, when $x=\frac{\pi}{2}$.

Integrate with respect to x the following functions from (44) to (70):—

(44) $\dfrac{x+1}{\sqrt{x-2}}$.

(45) $\dfrac{1}{x^2\sqrt{2+x^2}}$.

(46) $\dfrac{1}{x^2\sqrt{x^2-3x}}$.

(47) $\dfrac{1}{\sqrt{a^2x^2+x^4}}$.

(48) $\dfrac{1}{x\sqrt{1-x^3}}$.

(49) $\dfrac{1}{\sin^2 x-\cos^2 x}$.

(50) $\dfrac{x}{\sqrt{x^2+ax+b}}$.

(51) $\dfrac{1}{1+\cos^2 x}$.

(52) $\dfrac{1}{x(1+x^2)^2}$.

(53) $\dfrac{1}{(2+x^2)\sqrt{1+x^2}}$.

(54) $\dfrac{1}{1-\tan^2 x}$.

(55) $\dfrac{x-1}{\sqrt{x^2-x+3}}$.

(56) $\dfrac{1}{\sqrt{1-e^{2x}}}$.

(57) $\dfrac{1}{\cos x(3-\cos x)}$.

(58) $\dfrac{15-\cos 2x}{3+\cos 2x}$.

(59) $\dfrac{x+2}{x^3-1}$.

(60) $\dfrac{2+x}{\sqrt{1+x-x^2}}$.

(61) $\dfrac{2x^2-x-2}{(x^2+2)(x^2-3x+2)}$.

(62) $\dfrac{1+x}{(1+x^2)^3}$.

(63) $\dfrac{\mathrm{L}x+\mathrm{M}}{(ax^2+bx+c)^2}$.

(64) $\sin^3 x\cos^4 x$.

(65) $\dfrac{1}{\sin(x-a)\sin(x-b)}$.

(66) $\dfrac{\sin x}{\sin(x-a)\sin(x-b)}$.

(67) $\dfrac{1}{\sin(x-a)\sin(x-b)\sin(x-c)}$.

(68) $\dfrac{1}{x(a^2-x^2)^{\frac{3}{2}}}$.

(69) $\dfrac{(1+2x^{\frac{3}{4}})^{\frac{2}{3}}}{x^3}$.

(70) $\dfrac{1}{x^5(2-x^3)^{\frac{5}{3}}}$.

(71) Find a formula of reduction for $\displaystyle\int \frac{dx}{x^n\sqrt{a^2-x^2}}$, where n is a positive integer.

Find the values of the following definite integrals—

(72) $\displaystyle\int_0^1 \frac{x^3\,dx}{(x+1)(4-x^2)}$.

(73) $\displaystyle\int_0^2 \frac{\sqrt{5-2x}}{1+2x}\,dx$.

(74) $\displaystyle\int_0^a x^3(a^2+x^2)^{\frac{n}{2}}\,dx$.

(75) $\displaystyle\int_0^{\frac{1}{2}} x^2 \sin^{-1} x\,dx$.

(76) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$.

(77) $\displaystyle\int_0^{r\pi} \sin^{14} x \cos^8 x\,dx$.

(78) $\displaystyle\int_0^3 x^4\sqrt{3x-x^2}\,dx$.

(79) $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin^4 x\,dx$.

(80) $\displaystyle\int_0^1 e^{x^2} x^3\,dx$.

(81) $\displaystyle\int_0^{\pi} x^2 \cos^2 x\,dx$.

(82) $\displaystyle\int_0^{\pi} e^x \cos 4x\,dx$.

(83) $\displaystyle\int_0^{\pi} e^{-x} \cos^2 x\,dx$.

(84) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^5 x}$.

(85) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

(86) $\displaystyle\int_0^{\pi} (1+\cos x)^n\,dx$ where n is a positive integer.

(87) $\displaystyle\int_1^{\infty} \left(\frac{\log x}{x}\right)^n dx$, where n is a positive integer.

(88) $\displaystyle\int_0^6 \frac{x^3+12x^2+4x}{x^4-16}\,dx$.

(89) $\displaystyle\int_0^{\sqrt{2}} \frac{3x^4+x^3+11x^2+4x+8}{x^5-x^4+4(x^3-x^2+x-1)}\,dx$.

ANSWERS.

(1) $-2\sqrt{\dfrac{(2+x)^3}{2-x}}$. (2) $-m\dfrac{(1+\sqrt{1-x^2})^m}{x^{m+1}\sqrt{1-x^2}}$.

(3) $(a-x)^{3x^2}\left\{6x\log(a-x)-\dfrac{3x^2}{a-x}\right\}$.

(4) $x^2\dfrac{x^2+2ax+3a^2}{3(a+x)(a^2+x^2)(a^2+ax+x^2)}$.

(5) $\dfrac{b^2}{4}\cdot\dfrac{\sin(4x-2a)+2\sin(2x-a)}{\sqrt{a^2-b^2\cos^4\left(x-\dfrac{a}{2}\right)}}$.

(6) $a^{3x}\left\{3\log a\log(1-x^2)-\dfrac{2x}{1-x^2}\right\}$.

(7) $\cos 8x$.

(8) $$3x^2\log\sqrt{\left(1-\sin\frac{a}{x}\right)\left(1-\cos\frac{a}{x}\right)}+\frac{ax}{2}\left\{\frac{\left(\cos\frac{a}{x}-\sin\frac{a}{x}\right)\left(1-\cos\frac{a}{x}-\sin\frac{a}{x}\right)}{\left(1-\cos\frac{a}{x}\right)\left(1-\sin\frac{a}{x}\right)}\right\}.$$

(9) $2(\sec 4x-\tan 4x)$.

(10) $\dfrac{1}{1+x^2}$. (11) $\dfrac{1}{1+x^2}$.

(12) $\dfrac{\sin x}{1+\cos^2 x}$. (13) $\dfrac{x}{\sqrt{x^2-1}}$.

(14) $e^{ax}\left\{a\sin^{-1}2x\sqrt{1-x^2}+\dfrac{2}{\sqrt{1-x^2}}\right\}$.

(15) $\dfrac{1}{2\sqrt{1-x^2}}\cdot\dfrac{x+\sqrt{1-x^2}}{1-x\sqrt{1-x^2}}$. (16) $\dfrac{a}{(a-x)\sqrt{2ax-3x^2}}$.

(17) $\dfrac{6}{\cos 2x}$. (18) $2xa^{3x^2}\left\{3\log a.\cos^{-1}\dfrac{2x^2+1}{2x^2+3}-\dfrac{\sqrt{2}}{\sqrt{1+x^2}}\cdot\dfrac{1}{(2x^2+3)}\right\}$.

(19) $a\dfrac{\sin\dfrac{a}{x}-\cos\dfrac{a}{x}}{x^2\left(2+\sin\dfrac{2a}{x}\right)}$. (20) $(\tan^{-1}x)\ \{1+\tan^{-1}x\log(\tan^{-1}x)^{2x}\}$.

(21) $(\log x)^{x-1}\cos(\log x)^x\{1+\log x.\log\log x\}$.

(22) $(\sin\log x)^{x-1}\{\cos\log x+(\sin\log x)\log(\sin\log x)\}$.

(23) $\dfrac{2a^2}{(1-a^2x^2)}\left\{1-\dfrac{ax\cos^{-1}ax}{\sqrt{(1-a^2x^2)}}\right\}$.

(27) $\dfrac{y\cos^2 y-(1+x^2y^2)\sin y\cos y}{x(\sin^2 y+x^2y^2)}$. 8) $\dfrac{y\cos x-ye^{x+y}}{ye^{x+y}-1}$.

(29) $\log 2-\dfrac{x}{2}-\dfrac{x^2}{8}-\dfrac{x^3}{24}-\dfrac{x^4}{64}\ldots.$ (30) $1+\dfrac{x^2}{6}+\dfrac{7x^4}{360}\ldots.$

(31) $\dfrac{5}{112}$.

(32) $x\log x-h(1+\log x)+\dfrac{h^2}{2x}+\dfrac{h^3}{6x^2}+\dfrac{h^4}{12x^3}+\dfrac{h^5}{20(x+\theta h)^4}$,
where θ is a proper fraction.

(33) 128 is a maximum, 125 and 0 are minima values.

(34) $3(1-\log 3)$ is a minimum; there is no maximum value.

(36) $x=2n\pi$ gives a maximum, $x=(2n+1)\pi$ a minimum value, n being any integer. If $x=\cos^{-1}\dfrac{\sin a}{\sqrt{3}}$, the value is a minimum.

(37) The semivertical angle of the cone is (1) $\sin^{-1}\dfrac{\sqrt{3}}{3}$, (2) $\sin^{-1}\frac{1}{2}$.

(38) (1) The volume is $\dfrac{4\pi}{81}h^3\tan^2 a$.

(2) If $\sin a$ is greater than $\frac{1}{3}$ there is no maximum; if $\sin a$ is not greater than $\frac{1}{3}$ the vertical angle of the required cone is $\sin^{-1}(3\sin a)-a$.

(39) $\sqrt{10}$. (40) 1.

(41) 1.

(42) 1.

(43) $\frac{1}{2}$.

(44) $\frac{2}{3}\sqrt{(x-2)}(x+7)$.

(45) $-\frac{\sqrt{(2+x^2)}}{2x}$.

(46) $\frac{2(3+2x)\sqrt{(x-3)}}{27x^{\frac{3}{2}}}$.

(47) $\frac{1}{a}\log\frac{\sqrt{a^2+x^2}-a}{x}$.

(48) $\log\sqrt[3]{\frac{1-\sqrt{1-x^3}}{1+\sqrt{1-x^3}}}$.

(49) $\frac{1}{2}\log\tan\left(x-\frac{\pi}{4}\right)$.

(50) $\sqrt{(x^2+ax+b)}-\frac{a}{2}\log\left\{x+\frac{a}{2}+\sqrt{(x^2+ax+b)}\right\}$.

(51) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$.

(52) $\log\frac{x}{\sqrt{1+x^2}}-\frac{x^2}{2(1+x^2)}$.

(53) $\frac{1}{2\sqrt{2}}\log\frac{x+\sqrt{2+2x^2}}{x-\sqrt{2+2x^2}}$. Substitute $x=\tan y$, and the given integral reduces to $\int\frac{d\sin\theta}{2-\sin^2\theta}$.

(54) $\frac{x}{2}+\frac{1}{8}\log\frac{1+\sin 2x}{1-\sin 2x}$.

(55) $\sqrt{(x^2-x+3)}-\frac{1}{2}\log\{2x-1+2\sqrt{x^2-x+3}\}$.

(56) $\frac{1}{2}\log\frac{1-\sqrt{1-e^{2x}}}{1+\sqrt{1-e^{2x}}}$.

(57) $\frac{1}{6}\log\frac{1+\sin x}{1-\sin x}+\frac{1}{3\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan\frac{x}{2}\right)$.

The integral equals $\frac{1}{3}\int\left(\frac{1}{\cos x}+\frac{1}{3-\cos x}\right)dx$.

(58) $\frac{9}{\sqrt{2}}\tan^{-1}\frac{\tan x}{\sqrt{2}}-x$.

(59) $\log \dfrac{x-1}{\sqrt{1+x+x^2}} - \dfrac{1}{\sqrt{3}} \tan^{-1} \dfrac{2x+1}{\sqrt{3}}.$

(60) $\dfrac{5}{2} \cos^{-1} \dfrac{1-2x}{\sqrt{5}} - \sqrt{(1+x-x^2)}.$

(61) $\dfrac{1}{3\sqrt{2}} \tan^{-1} \dfrac{x}{\sqrt{2}} + \log \dfrac{\sqrt[3]{(x-1)(x-2)^2}}{\sqrt{x^2+2}}.$

(62) $\dfrac{x-1}{4(1+x^2)^2} + \dfrac{3x}{8(1+x^2)} + \dfrac{3}{8}\tan^{-1} x.$

(63) $-\dfrac{\mathrm{L}}{2a} \cdot \dfrac{1}{(ax^2+bx+c)} + \left(\mathrm{M} - \dfrac{\mathrm{L}b}{2a}\right)\left\{\dfrac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \right.$

$\left.\dfrac{4a}{(4ac-b^2)^{\frac{3}{2}}} \tan^{-1} \dfrac{2ax+b}{\sqrt{4ac-b^2}}\right\}.$

(See CHAP. VIII., § 5.)

(64) $\dfrac{5\cos^7 x - 7\cos^5 x}{35}.$

(65) $\dfrac{1}{\sin(a-b)} \log \dfrac{\sin(x-a)}{\sin(x-b)}.$

(66) $\dfrac{1}{\sin(a-b)} \left\{ \sin a \log \tan \dfrac{x-a}{2} - \sin b \log \tan \dfrac{x-b}{2} \right\}.$

Put the given expression into the form

$$\frac{1}{\sin(a-b)} \left\{ \frac{\sin a}{\sin(x-a)} - \frac{\sin b}{\sin(x-b)} \right\}.$$

(67) $\dfrac{1}{\sin(a-b)\sin(a-c)} \log \tan \dfrac{x-a}{2} + \dfrac{1}{\sin(b-a)\sin(b-c)} \log \tan \dfrac{x-b}{2}$

$+ \dfrac{1}{\sin(c-a)\sin(c-b)} \log \tan \dfrac{x-c}{2}.$